Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.
1. Find the leading term in the asymptotic expansion of the following integral, in the indicated limits

\[ \int_0^2 e^{\sqrt{t}} \exp \left[ \lambda (t - 1)^2 \right] \, dt. \]

(a) \( \lambda \to +\infty \)  
(b) \( \lambda \to -\infty \).

2. Consider the ODE

\[ x^2 y''(x) + x y'(x) - (3 + x^2) y(x) = 0. \]

(a) Classify all points, including \( x = \infty \), as ordinary, regular singular or irregular singular.

(b) Find the asymptotic behaviors of all solutions, in the limit \( x \to 0 \).

(c) Find the asymptotic behaviors of all solutions, in the limit \( x \to \infty \).

3. Consider the sum, for \( a > 0 \) and real \( \lambda \),

\[ S(\lambda; a) = \sum_{n=0}^{\infty} e^{-\lambda n} e^{-an^2}. \]

(a) If \( \lambda \to \infty \), with \( a \) fixed, find a two term asymptotic approximation to \( S \).

(b) If \( a \to 0 \), with \( \lambda > 0 \) fixed, find a one term asymptotic approximation to \( S \).

(c) If \( a \to 0 \), with \( \lambda = 0 \), find a two term asymptotic approximation to \( S \).

(d) If \( a, \lambda \to 0 \), such that \( \lambda = \sqrt{a} \beta \) with \( \beta \) fixed, find a one term asymptotic approximation to \( S \).
4. Consider the following eigenvalue problem, for $\lambda \to \infty$,

$$y''(x) + \lambda x^2 y(x) = 0, \quad 1 < x < 2; \quad y'(1) = 0, \quad y(2) = 0.$$

Find the asymptotic behavior of the large eigenvalues and the corresponding eigenfunctions. Do not normalize the eigenfunctions.

5. Find two-term asymptotic approximations, as $\epsilon \to 0^+$, for all roots $x = x(\epsilon)$ of the algebraic equation

$$\epsilon x^4 + x^3 - x^2 + \epsilon = 0.$$

6. Use the two-time method to analyze the following, for $\epsilon \to 0^+$,

$$y''(t) + 2\epsilon y'(t) + 4y(t) = 0,$$

$$y(0) = 1, \quad y'(0) = 0.$$

You must use the two-time method to receive credit.

7. Consider the following singularly perturbed boundary value problem:

$$\epsilon y''(x) + \beta xy'(x) + xy(x) = 0, \quad 0 < x < 1; \quad y(0) = A, \quad y(1) = B.$$

Use boundary layer theory to analyze the problem as $\epsilon \to 0^+$, for the cases

(a) $\beta = 1$ and (b) $\beta = -1$. 
8. Consider the following singularly perturbed boundary value problem, defined in the unit circle \( C = \{(x, y) : x^2 + y^2 < 1\} \)

\[
\varepsilon \Delta u + u_x + u_y = \varepsilon [u_{xx} + u_{yy}] + u_x + u_y = 0, \quad (x, y) \in C \\
u = f(\theta), \quad \text{when } r = \sqrt{x^2 + y^2} = 1.
\]

Construct an outer solution, locate any boundary/internal layers, and give their thickness. Give the ODE/PDE that applies in each layer, and solve it. Suggestion: first sketch the domain and the subcharacteristics.