• The Spring 2010 Mathematical Science Preliminary examination covers the areas of:
  – Mathematics of Fluid Dynamics
  – Linear and Nonlinear Waves
  – Computational Finance

• Students elect to answer questions in only two of these areas.

• There are 4 questions in each area; each question is worth 20 points. All questions in only two requested areas will be graded, but your score for the examination will be the sum of the scores of your best FIVE questions.

• Use a separate answer booklet for each area of the exam.
1. Find the complex potential of a single line vortex of strength $\Gamma$ in the sector

$$\Omega = \left\{ z : 0 \leq \arg z \leq \frac{\pi}{2} \right\}$$

with rigid walls. Find the equation of motion of the vortex.

2. Consider the Euler equations for the velocity $u(x, t)$ and the pressure $p(x, t)$:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = -\nabla p, \quad \nabla \cdot u = 0.$$ 

Derive an equation for the vorticity $\omega = \nabla \times u$ and prove that in two dimensions

$$\frac{\partial \omega}{\partial t} + (u \cdot \nabla)\omega = 0.$$ 

Obtain an equation for the stream function.

3. State and prove the Bernoulli streamline theorem.

4. Consider the Navier-Stokes equations on a torus

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u = \nu \Delta u - \nabla p, \quad x \in (\mathbb{R}/\mathbb{Z})^3,$$

$$\nabla \cdot u = 0.$$ 

Show that the total energy of every nontrivial smooth solution with zero mean decreases.
1. Consider the forced PDE
\[ u_t - u_{xxx} + 4u_x = f_x(x - ct) \]
when \( x \in \mathbb{R} \) and \( f(\xi) \) is a localized, bounded real function. Find steady solutions if
\begin{enumerate}
  \item \( c = 3 \),
  \item \( c = 5 \),
\end{enumerate}
applying a radiation condition at \( |x| \to \infty \) if necessary. In both cases, describe the solution. For localized solutions, find the decay rate and the frequency of any oscillations. For non-decaying solutions, what frequencies may be found at \( x = \pm \infty \)?

2. Consider
\[ u_t + uu_x = 0 \]
a conservation law for \( u \). Solve the initial value problem on \((x, t) \in (\mathbb{R}, \mathbb{R}^+)\) when
\[
u(x, 0) = \begin{cases} 
1 & x < 0 \\
2 & 0 < x < 1 \\
1 & x > 1 
\end{cases}
\]
As part of your solution, plot the profiles \( u(x, T) \) at \( T = 1, 2, \) and \( 3 \) as well as the characteristics in the \( x-t \) plane.

3. Consider the linear potential flow equations
\begin{align*}
\eta_t - \phi_z &= 0 \\
\phi_t + g\eta - \gamma \eta_{xx} &= 0 \\
\phi_{xx} + \phi_{zz} &= 0 \\
\phi_z &= 0
\end{align*}
which govern a free surface displacement \( \eta(x, t) \) and a potential \( \phi(x, z, t) \). Here \( g \) and \( \gamma \) are the gravity and surface tension constants respectively. The fluid lies above a flat bottom at \( z = -H \). Calculate the dispersion relation, \( \omega \), for a wave \( \eta = e^{ikx-i\omega t} \). Find the frequency of the slowest wave (smallest phase speed) in the deep water limit \((H \to \infty)\).

4. Consider the dispersive wave equation
\[ u_t - \mathcal{H}u_{xx} - \frac{3}{2} \mathcal{H}u + \epsilon uu_x = 0 \]
with \( \epsilon \ll 1 \) and \( \mathcal{H} \) the Hilbert transform \((\mathcal{H} = -i\text{sign}(k))\).
i Show that the linear equation ($\epsilon = 0$) supports waves which travel at speed $c = 5/2$, and are supported at two frequencies $u = e^{i k_1 x} + \beta e^{i k_2 x} + ^*$, where $^*$ denotes complex conjugate. In the linear solution $\beta$ is an arbitrary constant. Calculate $k_1, k_2$ explicitly.

ii Calculate $\beta$ in the weakly nonlinear solution ($\epsilon \neq 0$) by expanding

$$u = e^{i k_1 x} + \beta e^{i k_2 x} + ^* + \epsilon u_1(x,t) + \epsilon^2 u_2(x,t) + ...$$

$$c = 5/2 + \epsilon c_1 + \epsilon^2 c_2 + ...$$

and computing the solvability conditions at $O(\epsilon)$. 

4
1. Consider two portfolios:
   - \( \pi_A \): One European call option plus \( Ee^{-rT} \) cash (invested in a bank).
   - \( \pi_B \): One European put option plus one unit of the underlying asset.

(a) By valuing \( \pi_A \) and \( \pi_B \) at expiry \( (t = T) \) and appealing to the No Arbitrage Principle (be quantitative and explicit) deduce “Put–Call Parity”.

(b) By modifying the portfolios above find lower bounds on the values of both a European call and a European put.

2. Consider the Heat Equation:
   \[
   \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad -L < x < L, \quad t > 0, \\
   u(x, 0) = f(x), \quad -L < x < L, \\
   u(-L, t) = u(L, t) = 0, \quad t > 0.
   \]

The \( \theta \)-scheme to numerically approximate its solutions is
\[
-\theta \beta v_{j+1}^{n+1} + (1 + 2\theta \beta)v_j^{n+1} - \theta \beta v_{j-1}^{n+1} = (1 - \theta)\beta v_{j+1}^n + (1 - 2(1 - \theta)\beta)v_j^n + (1 - \theta)\beta v_{j-1}^n,
\]
where \( \beta = \Delta t/(\Delta x)^2 > 0, \ 0 \leq \theta \leq 1. \)

(a) Show that this scheme is consistent with the Heat Equation for any \( 0 \leq \theta \leq 1. \)

(b) Show that this scheme is unconditionally stable if \( \theta = 3/4. \)

(c) Show that this scheme is convergent if \( \theta = 3/4. \)

3. Find a similarity solution of the Heat Equation:
   \[
   \frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u, \quad x, t > 0, \\
   u(x, 0) = 0, \quad x > 0, \\
   u(0, t) = 1, \quad t > 0, \\
   u \to 0 \quad x \to \infty.
   \]

4. Show that early exercise of an American call option is never optimal. Be quantitative. Using explicit formulas with precise monetary amounts show how, in certain cases, one could make a risk–free profit in the event of early exercise.