Preliminary Examination in Algebra

Spring, 2000

Five correct solutions earn a grade of 1, four earn a 2 and three earn a 3.

1. Let $F$ be the field of fractions of a unique factorization domain $R$. Let $F^*$ denote the multiplicative group $F - \{0\}$, and let $R^*$ denote the subgroup of $F^*$ consisting of all invertible elements of $R$. Prove that the quotient group $F^*/R^*$ is a free abelian group.

2. Let $R$ be a commutative ring which has only one maximal ideal, denoted $\mathcal{M}$. Let $L$ be a free module of finite rank over $R$, and let $f$ be an endomorphism of $L$, i.e. a module homomorphism from $L$ to itself. Let $\tilde{f}$ denote the unique endomorphism of the $R/\mathcal{M}$-module $L/\mathcal{ML}$ such that the diagram

$$
\begin{array}{ccc}
L & \xrightarrow{f} & L \\
\downarrow & & \downarrow \\
L/\mathcal{ML} & \xrightarrow{\tilde{f}} & L/\mathcal{ML}
\end{array}
$$

commutes. Prove that if $\tilde{f}$ is an automorphism of $L/\mathcal{ML}$ then $f$ is an automorphism of $L$.

3. Let $G$ be a group of order 27. Prove that the automorphism group of $G$ cannot contain an element of order 5.

4. Let $K$ be a field and let $f \in K[X]$ be a polynomial of degree $> 0$. Prove that $f$ has a root in some extension of $K$.

5. A field of characteristic $p$ is called perfect if each element is a $p$-th power.
   (i) Show that every finite field is perfect.
   (ii) Show that any algebraic extension of a perfect field is separable.

6. Let $G$ be a group which has a proper subgroup of finite index. Prove that $G$ has a proper normal subgroup of finite index.

7. Determine all groups of order 52 up to isomorphism. Give the order of the center in each case.

8. Determine, with proof, the isomorphism type of the Galois group of the polynomial $X^3 - 2$ over the field $\mathbb{Q}$ of rational numbers. Describe explicitly, with proof, all the subfields of the splitting field of this polynomial.