Instructions: There are 8 problems on this exam, each of equal value. Five correct answers earns a score of one, four a score of two, and three a score of three. (Note: all rings are assumed to have a unit element!)

1. A group $G$ is metabelian if $G'$ is not trivial, but $(G')'$ is trivial, where $G'$ denotes the commutator subgroup of $G$. Prove that $G$ is metabelian if and only if $G$ is a nonabelian group with an abelian normal subgroup $A$ such that $G/A$ is abelian.

2. Let $p$ and $q$ be distinct prime numbers.
   a. Prove that a group of order $pq$ is solvable.
   b. Prove that a group of order 30 is solvable.

3. Let $R$ be a ring with unit.
   a. Prove that the intersection of all primitive two-sided ideals in a ring $R$ is the same as the intersection of all maximal right ideals.
   b. Suppose that $M \subset R$ is a maximal (two-sided) ideal. Prove that $M$ is primitive.

4. Let $n > 1$ be an integer and let $\zeta_n = e^{2\pi i/n}$. Let $\Phi_n$ be the minimal polynomial of $\zeta_n$ over $\mathbb{Q}$.
   a. Prove that $\Phi_n$ has integer coefficients.
   b. Let $\mathbb{F}_q$ be a finite field with $q = p^r$ elements for some prime number $p$. Suppose that $p$ does not divide $n$. Let $\overline{\Phi}_n(X)$ be the reduction of $\Phi_n(X)$ mod $p$, which we view as an element of $\mathbb{F}_q[X]$. Prove that $\overline{\Phi}_n(X)$ has a root in $\mathbb{F}_q$ if and only if $n$ divides $q - 1$.

5. Let $G$ be the group $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and let $H$ be the subgroup generated by the vectors $(2, 4, 15), (-13, -2, -19)$, and $(7, 2, 11)$. Determine the structure of the group $G/H$.

6. Let $F$ be a field, $f(x)$ an irreducible polynomial over $F$, and $K$ a finite normal extension of $F$. Suppose that $g$ and $h$ are irreducible elements of $K[x]$ such that $f = gh$. Show that there exists an automorphism $\sigma$ of $K$, fixing $F$, such that $g = \sigma(h)$.

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7. Give examples (with proof) of:
   a. A ring $R$ which is not Noetherian.
   b. A ring $R$ which is Noetherian, but not Artinian.
   c. A ring $R$ and a right $R$-module $M$ which is not projective.
   d. A ring $R$ and a right $R$-module $M$ which is projective, but not free.

8. Prove that a 72 degree angle can be trisected with ruler and compass, but a 60 degree angle cannot.