Instructions: There are 8 problems on this exam, each of equal value. Five correct answers earns a score of one, four a score of two, and three a score of three. (Note: all rings are assumed to have a unit element!)

1. Let $G$ be a finite group of order $m$, and let $H$ be a normal cyclic subgroup of $G$ of prime order $p$. Suppose that $m$ is relatively prime to $p - 1$. Prove that $H$ belongs to the center of $G$.

2. Let $G$ be a finite group and let $X$ be a finite set on which $G$ acts. For $g \in G$, let $\chi(g)$ be the number of fixed points of $g$ on $X$. Let $n$ be the number of orbits of $G$ on $X$. Prove that
   \[ n = \frac{1}{|G|} \sum_{g \in G} \chi(g). \]

3. Let $E/F$ be a field extension and let $L$ be the subset of $E$ consisting of elements which are algebraic over $F$. Prove that $L$ is a field.

4. Let $E$, $L$, and $F$ be fields with $F \subset L \subset E$. Suppose that $E$ is normal over $F$. Prove that $E$ is normal over $L$, and give an example to show that $L$ need not be normal over $F$.

5. Let $f$ be an irreducible polynomial of degree $d$ over the finite field of order $p$. Let $F$ be a field with $p^n$ elements. Prove that $f$ has a root in $F$ if and only if $d$ divides $n$.

6. Let $R$ be a Noetherian ring, and let $M$ be a finitely generated right $R$-module. Prove that every right submodule of $M$ is finitely generated.

7. Let $\zeta$ be a primitive $13^{th}$ root of unity.
   A. Prove that $E = \mathbb{Q}(\zeta)$ has a unique subfield $F$ of degree 3 over $\mathbb{Q}$ and a unique subfield $L$ of degree 4 over $\mathbb{Q}$.
   B. Prove that $F = \mathbb{Q}(\alpha)$, where $\alpha = \zeta + \zeta^5 + \zeta^8 + \zeta^{12}$.

8. Let $R$ be a ring (with unity, as always.)
   A. Let $P$ be a right $R$-module. Prove that the following two conditions are equivalent:
      1. Given right $R$ modules $M$ and $N$, a surjective map $\phi : M \to N$, and a map $\pi : P \to N$, there is a map $f : P \to M$ such that $\phi \circ f = \pi$.
      2. There exists a right $R$-module $Q$ such that $P \oplus Q$ is free.
   B. Suppose every simple right $R$-module is free. Prove that $R$ is a division ring.