Preliminary Exam in Real and Complex Analysis

Spring 2006,

Answer 5 out of the following 8 questions. Two of them should be from one part of the exam and three from the other.

Real Analysis

1. Let $V$ be a linear subspace of a Hilbert space $H$. Prove that the closure $\overline{V}$ equals $(V^\perp)^\perp$.

2. For a sequence $\{f_n\}$ of measurable functions on $[a, b]$ with
   \[ \sum_{n=1}^{\infty} \left( \int_a^b |f_n(x)| \, dx \right) < \infty \]
   prove that $\lim_{n \to \infty} f_n(x) = 0$ for almost every $x \in [a, b]$.

3. Let $f$, $g$, and $f_n$ be $L^1$-functions on some probability space $(X, \mu)$. Assume that $|f_n(x)| \leq |g(x)|$ for all $x$ and that $f_n \to f$ in measure. Prove that $f_n \to f$ in $L^1(X, \mu)$.

4. Let $E \subset [a, b]$ be a Lebesgue measurable set with $m(E) > 0$. Using outer measure prove that for any $\alpha < 1$ there exists an interval $I \subset [a, b]$ with $m(E \cap I) > \alpha \cdot m(I)$. Give an example of a set $E \subset [0, 1]$ as above, so that $m(E \cap I) < m(I)$ for every non-trivial interval $I$.

Complex Analysis

1. Let $(a_n)_{n \geq 1}$ be any bounded sequence of complex numbers. Prove that the series
   \[ \sum_{n=1}^{\infty} \frac{a_n}{n^\sigma} \]
   converges for every $z$ in the half-plane $\Re z > 1$, and defines a function which is holomorphic in this half-plane.

2. Suppose $D$ is the open unit disc and $f : D \to D$ is analytic. Suppose $f'(0) = z_0$. Show that $|f'(z_0)| \leq 1 - |z_0|^2$.

3. Suppose $\gamma$ is a simple closed curve in the complex plane and $\Omega$ is the interior of $\gamma$. For any continuous function $\phi : \gamma \to \mathbb{C}$, let $f_\phi : \Omega \to \mathbb{C}$ be defined by
   \[ f_\phi(z) = \int_{\gamma} \frac{\phi(\zeta)}{\zeta - z} \, d\zeta. \]
Show that
\[ \{ f_\phi : |\phi(\zeta)| \leq 1, \forall \zeta \in \gamma \} \]
is a normal family in \( \Omega \).

4 Let \( \Omega \) be a domain that contains the half plane \( H \) defined by \( \text{Re} z \geq 0 \). Let \( F(z) \) be a holomorphic function defined on \( \Omega \) with the property that for some \( M \) and all \( z \in H, z \neq 0 \),
\[ |F(z)| \leq \frac{M}{|z|^2}. \]

Let \( (t_n) \) be any sequence in \( H \) with \( |t_n| \to \infty \) and \( \gamma_n \) a smooth curve in \( H \) from 0 to \( t_n \). Show that the limit
\[ \lim_{n \to \infty} \int_{\gamma_n} F(z) \, dz \]
exists.