Analysis Preliminary Exam

May 15, 2012

Answer 5 out of the following 8 questions

Grading scheme:
3+2 (or 2 +3) correct solutions for the grade of 1, 2+2 for 2, 2+1 (or 1 + 2) for 3.

R1. Let \( \{f_n\}_{n=1}^{\infty} \) be a sequence of nonnegative measurable functions that converges to \( f \) in measure. Show that
\[
\int f \leq \liminf_{n \to \infty} \int f_n.
\]

R2. Let \((X, \mathcal{M}, \mu)\) be a measure space with \( \mu(X) < \infty \) and \( f \in L^\infty(X) \). Prove that
\[
\lim_{n \to \infty} \left( \int_X |f|^n \, d\mu \right)^{\frac{1}{n}} = \|f\|_\infty.
\]

R3. Let \( \nu \) be a signed measure on \((X, \mathcal{M})\). Show that
\[
|\nu|(E) = \sup \left\{ \left| \int_E f \, d\nu \right| : |f| \leq 1 \right\}
\]
for every \( E \in \mathcal{M} \).

R4. Let \((X, \mathcal{M}, \mu)\) be a finite measure space and \( \{A_n\}_{n=1}^{\infty} \subset \mathcal{M} \). Prove that
\[
\mu(\liminf_{n \to \infty} A_n) \leq \liminf_{n \to \infty} \mu(A_n) \leq \limsup_{n \to \infty} \mu(A_n) \leq \mu(\limsup_{n \to \infty} A_n).
\]

C1. Suppose \( f(z) \) is analytic on \( \{z \in \mathbb{C} : |z| > 1\} \) such that \( \lim_{z \to \infty} \frac{f(z)}{z} = 0 \). Show that there is \( c \in \mathbb{C} \) with \( \lim_{z \to \infty} f(z) = c \).

C2. Prove that \( \sum_{n=1}^{\infty} \frac{1}{(z-n)^2} \) defines a meromorphic function on \( \mathbb{C} \).

C3. Find all analytic functions \( f(z) \) on the unit disc such that \( f'(\frac{1}{n}) = 2f(\frac{1}{n}) \) for all integers \( n > 1 \). Prove your answer is correct.

C4. Suppose \( h(t) \) is a complex valued function on \([0, 1]\) such that \( \int_0^1 |h(t)| \, dt < \infty \). Show that \( f(z) = \int_0^1 \frac{h(t)}{z-t} \, dt \) defines an analytic function on \( \mathbb{C} - [0,1] \).