The best FIVE answers will determine your grade.

1. Prove the Cauchy-Goursat theorem: if $f$ is a holomorphic function on an open set $U \subseteq \mathbb{C}$, and $R$ is a standard rectangle contained in $U$, then

$$\int_{\partial R} f = 0.$$  

(A standard rectangle is a set of the form $R = \{x + iy : a \leq x \leq b, c \leq y \leq d\}$.)

You may ASSUME the following (elementary) fact:

**Lemma.** If $f$ is holomorphic on $U$ then for each $z_0 \in U$ and each $\epsilon > 0$, there is a $\delta > 0$ such that for every standard rectangle $r$ with $z_0 \in r \subseteq U$ such that the diameter $\rho$ of $r$ is $< \delta$, we have

$$|\int_{\partial r} f| < \epsilon \rho^2.$$  

2. Suppose that $f$ is an entire function, and that there exist constants $C$ and $R$, and an integer $n > 0$, such that

$$|f(z)| \leq C|z|^n$$

for every $z$ with $|z| \geq R$. Prove that $f$ is a polynomial function of degree at most $n$.

3. Suppose that $(f_n)_{n \geq 1}$ is a sequence of holomorphic functions defined on an open set $U \subseteq \mathbb{C}$, which converges uniformly on every compact subset of $U$ to a function $f : U \to \mathbb{C}$. Prove that $f$ is holomorphic on $U$.

4. Use Liouville’s theorem to show that if $f(z)$ is a non-constant entire function then $f$ takes arbitrarily large real values on $\mathbb{C}$.

5. Evaluate

$$\int_{\gamma} \frac{z + 3}{(\cos z) - 1} \, dz$$

where $\gamma$ denotes the circle of radius 6 about 0, with the counterclockwise orientation.

6. Let $Q$ denote the first quadrant in the complex plane, consisting of all points $z$ such that $\text{Re}(z) > 0$ and $\text{Im}(z)$ are both strictly positive. Give an explicit formula for a conformal map of the $Q$ onto the unit disk $|z| < 1$. 

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7. Suppose that \( u \) is a harmonic function defined on an open set \( \Omega \subset \mathbb{C} \), and that \( \Omega \) contains a closed disk in \( D \) with center \( z_0 \) and radius \( R \). The mean value property of \( u \) is

\[
u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta.
\]

Under the ASSUMPTION that \( u \) is the real part of a holomorphic function on \( \Omega \), deduce the mean value property from Cauchy’s integral formula.

8. An open set \( U \subset \mathbb{C} \) is said to be simply connected if every cycle in \( C \) is homologous to 0. Prove that if \( U \) is simply connected then every holomorphic function \( f \) on \( U \) has a primitive. (Hint: use the general form of Cauchy’s theorem to show that for a piecewise smooth path \( \gamma \) in \( U \), the integral

\[
\int_{\gamma} f(z) \, dz
\]

depends only on the endpoints of \( \gamma \) and the function \( f \).)