REAL ANALYSIS PRELIMINARY EXAM
AUGUST 2 2005

There are 8 questions on two pages. Your grade will be based on the best 5 solutions.

Q1 Let \((X, \mathcal{M}, \mu)\) be a finite measure space, and suppose that \(f\) is a non-negative integrable real valued function on \(X\). For \(A \in \mathcal{M}\), define
\[
\lambda(A) := \int_A f \, d\mu
\]

a. Show that \(\lambda\) is a measure on \((X, \mathcal{M})\), and that \((X, \mathcal{M}, \lambda)\) is a finite measure space.

b. Show that if \(g\) is a non-negative measurable function on \(X\), then
\[
\int_X g \, d\lambda = \int_X gf \, d\mu.
\]

Q2 Let \((X, \mathcal{M}, \mu)\) be a measure space, and suppose that \(\{f_n\}\) is a sequence of real valued measurable functions on \(X\).

a. Define: “The sequence \(\{f_n\}\) converges to 0 in measure”.

b. Assuming that \((X, \mathcal{M}, \mu)\) is a finite measure space, show that there is a subsequence \(\{f_{n_k}\}\) of \(\{f_n\}\) which converges to 0 \(\mu\)-almost everywhere.

Q3

a. State the monotone convergence theorem.

b. Assuming the monotone convergence theorem, prove:

**Fatou's Lemma.** Suppose that \((X, \mathcal{M}, \mu)\) is a measure space, and \(\{f_n\}\) a sequence of non-negative functions in \(L^+(X)\), (i.e. measurable functions \(X \rightarrow [0, \infty]\)). Then:
\[
\int_X \liminf_n f_n \, d\mu \leq \liminf_n \int_X f_n \, d\mu
\]

c. Give an example of a sequence where the inequality is strict.
Q4 Let \((a, b], \mathcal{L}, \mu)\) be the measure space consisting of an interval on the real line equipped with Lebesgue measure. Prove that the space \(C([a, b])\) of continuous (complex valued) functions is dense in \(L^1([a, b])\).

Q5 (The Riemann-Lebesgue lemma.)
   a. Show that if \(f \in L^2([0, 2\pi])\), then
      \[
      \lim_{k \to \infty} \int_0^{2\pi} f(x) \cos(kx) \, dx = \lim_{k \to \infty} \int_0^{2\pi} f(x) \sin(kx) \, dx = 0.
      \]
   b. Show that this is also true if \(f \in L^1([0, 2\pi])\). You may use the the result of Q4.

Q6 Prove directly from the definitions that if \(f\) is absolutely continuous on the interval \([a, b]\), then \(f\) is of bounded variation on \([a, b]\).

Q7 Let \(E \subset \mathbb{R}^n\) be a Lebesgue measurable set, and suppose that \(f \in L^1(E) \cap L^2(E)\).
   a. Show that \(f \in L^p(E)\) for every \(p\) with \(1 \leq p \leq 2\).
   b. Compute \(\lim_{p \to 1^+} ||f||_p\).

Q8
   a. Prove that if \(p > 0\), \(q > 0\) and \(1/p + 1/q = 1\), then for any non-negative reals \(a\) and \(b\), \(ab \leq a^p/p + b^q/q\).
   b. State and prove Hölder's inequality for a pair of measurable functions on a measure space \((X, \mathcal{M}, \mu)\).