Show your work and justify your answers. The score for this exam will be based on the three best answers from each section. Two perfect answers from each section are sufficient for a best possible score of 1 on the prelim.

**Algebraic Topology**

**T1.** Show that there are no retractions $r : X \to A$ in the following cases:

(a) $X = S^1 \times D^2$ and $A$ being the boundary torus $S^1 \times S^1$.
(b) $X$ the Möbius band and is its boundary circle.

**T2.** The space $X$ is obtained by gluing a sphere $S^2$ with the torus $S^1 \times S^1$ via a homeomorphism that identifies of the equator in $S^2$ with the circle $\{x_0\} \times S^1$ in the torus. Compute

(a) The fundamental group $\pi_1(X)$.
(b) The homology $H_*(X)$.

**T3.** Let $X = \mathbb{R}P^2 \vee S^1$ - the projective plane with a circle attached to it at one point.

(a) Compute the fundamental group $\pi_1(X)$.
(b) Describe the universal cover $\tilde{X}$ of $X$.

**T4.** Let $L \subset S^3$ be a link with tree components, i.e.

$$L = f_1(S^1) \sqcup f_2(S^1) \sqcup f_3(S^1)$$

disjoint images of a circle under three embeddings $f_i : S^1 \to S^3$, $(i = 1, 2, 3)$. We assume that $K_i = f_i(S^1)$ have small disjoint neighborhoods $N_i$ homeomorphic to $D^2 \times S^1$ with $K_i$ corresponding to $\{0\} \times S^1$. Compute the homology $H_*(S^3 \setminus L)$. 

1
Differentiable Manifolds

G1. Let $\mathbb{R}^{2m}$ for $m \in \mathbb{Z}$ be the real projective space of even dimension.

(a) Define what an orientation on a manifold $M$ is.
(b) Show that $M = \mathbb{R}^{2m}$ is not orientable.
(c) Show that $M = f^{-1}(c)$ is orientable for $f : \mathbb{R}^{n+1} \to \mathbb{R}$ and $df(a) \neq 0$ when $a \in f^{-1}(c)$.

G2. Let $M$ be a smooth manifold of dimension $n$.

(a) Define the $p$-th De Rahm Cohomology $H^p(M)$ of $M$.
(b) Show that $H^p(\mathbb{R}^n) = 0$ for $p > 0$.
(c) Show that $H^1(S^1) \cong \mathbb{R}$.

G3. Consider $V$ a real vector space of dimension $n$.

(a) Give the definition of the exterior algebra $\Lambda^* V$.
(b) If $\alpha \in \Lambda^p V$ and $\beta \in \Lambda^q V$, find a formula for $\alpha \wedge \beta$, and prove that the formula is valid.
(c) Find $\dim \Lambda^i V$, for $1 \leq i \leq n$.

G4. Let $M$ be an $n$-dimensional oriented compact manifold with boundary $\partial M$.

(a) State Stokes theorem.
(b) Recall that a retract of $M$ onto a subset $N \subset M$ is a continuous map $f : M \to N$ such that $f(n) = n$ for all $n \in N$. Show that there is no smooth retract $M \to \partial M$.
(c) Show that the group $SL(2, \mathbb{R})$ is a smooth manifold.