Algebraic Topology Preliminary Exam
January, 2009

Do any five of the following eight problems. Show your work and justify your answers.

(1) In the commutative diagram of abelian groups

\[
\begin{array}{ccccccc}
A & \xrightarrow{i} & B & \xrightarrow{j} & C & \xrightarrow{k} & D & \xrightarrow{\ell} & E \\
\downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\varepsilon} \\
A' & \xrightarrow{i'} & B' & \xrightarrow{j'} & C' & \xrightarrow{k'} & D' & \xrightarrow{\ell'} & E'
\end{array}
\]

the rows are exact. Show that if \(\beta\) and \(\delta\) are onto and \(\varepsilon\) is one-to-one, then \(\gamma\) is onto.

(2) Let \(M\) be the Möbius band with boundary \(S^1\). Let \(X\) be the space obtained by attaching \(M\) to \(\mathbb{R}P^2\) by a homeomorphism from the boundary of \(M\) to the standard \(\mathbb{R}P^1\) in \(\mathbb{R}P^2\).

(a) [25\%] Calculate \(\pi_1(X)\).

(b) [75\%] Calculate \(H_\ast(X; \mathbb{Z})\).

(3) (a) Show any map \(f: \mathbb{R}P^2 \to S^1 \times S^1\) is nullhomotopic.

(b) Show there is an essential map in the other direction.

(4) Show any map \(f: S^{m+n} \to S^m \times S^n\), where \(m, n > 0\), induces the zero map on \(H^{m+n}\) and \(H_{m+n}\).

(5) Compute the homology of \(\mathbb{R}P^5 \times \mathbb{R}P^3\) with coefficients in \(\mathbb{Z}\) and \(\mathbb{Z}/2\).

(6) There is a closed 3-manifold \(L^3\) with \(\pi_1(L^3) = \mathbb{Z}/3\mathbb{Z}\).

(a) [25\%] Show \(L^3\) is orientable.

(b) [75\%] Assuming (a), calculate \(H_\ast(L^3; \mathbb{Z})\).

(7) Using the natural isomorphism \(H_2(S^2 \times \mathbb{C}P^2) \cong H_2(S^2) \oplus H_2(\mathbb{C}P^2)\) for integral homology, let \(f: S^2 \times \mathbb{C}P^2 \to S^2 \times \mathbb{C}P^2\) be a map such that \(f_2: H_2(S^2) \oplus H_2(\mathbb{C}P^2) \to H_2(S^2) \oplus H_2(\mathbb{C}P^2)\) gives \(f_2(a, b) = (3a, 2b)\).

(a) [75\%] Find the integral cohomology ring \(H^\ast(S^2 \times \mathbb{C}P^2)\) and find \(f^\ast: H^\ast(S^2 \times \mathbb{C}P^2) \to H^\ast(S^2 \times \mathbb{C}P^2)\).

(b) [25\%] Show that \(f\) must have a fixed point.

(8) Let \(X\) be a CW complex with cells as follows: 0-cell \(a\), 1-cells \(b, c\), 2-cells \(d, e\) and 3-cell \(f\). Use these same names for the generators in the cellular chain complex with boundaries \(\partial b = 0 = \partial c; \partial d = 4c; \partial e = 2c; \partial f = 3d - 6e\). Find the \(\mathbb{Z}/3\) homology and the \(\mathbb{Z}/2\) cohomology.