Do any five of the following eight problems. Show your work and justify your answers.

1. Let $\Delta^3$ be the standard 3-simplex, and let $K \subset \Delta^3$ be the union of a single 2-simplex with the 1-skeleton of $\Delta^3$. Calculate the fundamental group $\pi_1(K,v)$ for a vertex $v \in K$.

2. Let $X = S^2 \cup C^2$ be the union of the sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ with the infinite cylinder $C^2 = \{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 = 1/4\}$.
   (a) Find the homology $H_*(X;\mathbb{Z})$.
   (b) Find the local homology $H_*(X,X-p;\mathbb{Z})$ at a point $p \in S^2 \cap C^2$.

3. For the real projective plane $P^2$ and Klein bottle $K^2$, calculate the groups $H_*(P^2 \times K^2;\mathbb{Z})$ and $H^*(P^2 \times K^2;\mathbb{Z})$.
   (You may use your knowledge of $H_*(P^2;\mathbb{Z})$ and $H_*(K^2;\mathbb{Z})$.)

4. Let $M$ be a closed $n$-manifold such that $H_k(M;\mathbb{Z}_2) \neq 0$ for some $k$ with $0 < k < n$. Prove that there can be no map $f : S^n \to M$ with $f_* : H_n(S^n;\mathbb{Z}_2) \cong H_n(M;\mathbb{Z}_2)$.

5. Let $f : C_* \to D_*$ be a chain map between chain complexes $C_*$ and $D_*$. Define $\{E_*,\partial\}$ by $E_n = C_{n-1} \oplus D_n$ and $\partial(x,y) = (\partial x, \partial y + (-1)^n f(x))$ for $(x,y) \in E_n$. Define $\mu : D_* \to E_*$ by $\mu(y) = (0,y)$, and define $\nu : E_* \to C_{*-1}$ by $\nu(x,y) = x$.
   (a) Show that $\{E_*,\partial\}$ is a chain complex.
   (b) Show that $\mu$ and $\nu$ are chain maps and that $0 \to D_* \xrightarrow{\mu} E_* \xrightarrow{\nu} C_{*-1} \to 0$ is a short exact sequence of chain complexes.
   (c) Calculate the connecting homomorphism in homology.

6. Let $P^3 \# P^3$ be the connected sum of the real projective space $P^3$ with itself.
   (a) Find $\pi_1(P^3 \# P^3)$ using the Van Kampen theorem and your knowledge of $\pi_1 P^3$.
   (b) Prove that each map $f : P^3 \# P^3 \to S^1$ is null-homotopic.

7. Show that the Euler characteristic of a simply connected closed 4-manifold is positive.

8. (a) Find the degree of a map $f : S^n \to S^n$ with no fixed points.
   (b) Suppose that a finite group $G$ acts freely on $S^n$. Prove that if $G$ has order $\geq 3$, then $n$ must be odd.