Algebraic topology Prelim
May 15, 2002

Do any 5 of the following problems.

1) Show that $S^1 = CP^1 \subset CP^2$ is not a retract of $CP^2$.

2) Let $f : S^1 \times S^1 \to S^1 \times S^1$. Let $u_i = p_i^*(u) \in H^1(S^1 \times S^1)$ where $u \in H^1(S^1)$ is a generator and $p_i : S^1 \times S^1 \to S^1$ is the projection onto the $i$th factor for $i = 1, 2$. Suppose

$$
\begin{align*}
  f^*(u_1) &= au_1 + bu_2 \\
  f^*(u_2) &= cu_1 + du_2
\end{align*}
$$

for integers $a, b, c, d$.

   a) Calculate the degree of $f$ (deg $f$ is defined by $f^*(\mu) = (\deg f) \mu$ for $\mu \in H^2(S^1 \times S^1)$).

   b) Calculate the Lefshetz number $\tau(f)$ and show that if $(1 - a)(1 - d) \neq bc$, $f$ has a fixed point.

3) Let $f_\#: C_\# \to D_\#$ be a chain map. Define a chain complex $K$ by $K_n = D_n \oplus C_{n-1}$ and

$$
\partial(d, c) = (\partial d + (-1)^n f_\#(c), \partial c)
$$

for $(d, c) \in K_n$. Then there is a short exact sequence of chain complexes:

$$
0 \to D_n \xrightarrow{i} K_n \xrightarrow{j} C_{n-1} \to 0
$$

where $i(d) = (d, 0)$ and $j(d, c) = c$. Calculate the connecting homomorphism:

$$
H_{n-1}(C) \xrightarrow{\partial} H_{n-1}(D).
$$

4) Show that a compact simply connected 4 manifold has positive Euler characteristic.

5) Let $S^\infty = \bigcup_{n=0}^\infty S^n$ with the weak topology. $S^\infty$ is a CW complex with two $n$-cells in each dimension $n \geq 0$. Show that $S^\infty$ is contractible.
6) Let $\Sigma X = C^+X \cup_X C^-X$ be the union of two cones on $X$ with a common base. Show that $\tilde{H}^i(\Sigma X) \cong \tilde{H}^{i-1}(X)$. Does this give an isomorphism between the cohomology rings of $X$ and $\Sigma X$ with a shift in dimension?

7) We describe $S^1 \vee S^1$ with two circles oriented with labels $a$ and $b$. Consider the covering space described as follows

\[ C: \quad \begin{array}{ccc}
\bullet & \xrightarrow{b} & \circ \\
\circ & \xrightarrow{a} & \circ \\
\circ & \xrightarrow{x_0} & \circ \\
\circ & \xrightarrow{x_1} & \circ \\
\bullet & \xrightarrow{b} & \circ \\
\end{array} \]

Calculate the images $\pi_1(C, x_0) \rightarrow \pi_1(S^1 \vee S^1, \ast)$

$\pi_1(C, x_1) \rightarrow \pi_1(S^1 \vee S^1, \ast)$.

Are they the same subgroup? Are they conjugate subgroups?

8) Let $f : S^1 \times B^2 \rightarrow S^3$ be an embedding, $\Sigma = f(S^1 \times 0)$ and $V = S^3 - \Sigma$ ($\Sigma$ is a knot with a tubular neighborhood). Use the Mayer-Vietoris sequence to calculate $H_1(V; \mathbb{Z})$. 