Algebraic Topology Preliminary Exam

May 2, 1997

Do any five of the following eight problems.

1. Let \( f : A \to B, \ g : B \to C, \) and \( h : C \to D \) be maps such that \( g \circ f \) and \( h \circ g \) are homotopy equivalences. Prove that \( f, g, \) and \( h \) are homotopy equivalences. Hint: consider \( g \) first.

2. If \( M^3 \) is a compact, connected, orientable 3-manifold with \( H_1(M^3; \mathbb{Z}) = 0 \), show \( M \) has the homology of \( S^3 \).

If \( M^3 \) is nonorientable, show \( H_1(M^3; \mathbb{Z}) \) is infinite.

3. Let \( f : S^n \to S^n \) and let \( a : S^n \to S^n \) be the antipodal map.
   (i) If \( f(x) \neq x \) for all \( x \), then \( f \simeq a \).

   If \( f(x) \neq a(x) \) for all \( x \), then \( f \simeq 1_{S^n} \).

   (ii) If \( g : RP^n \to RP^n \), \( n \) even, then \( g \) has a fixed point.

4. The existence of a continuous, surjective map \( f : I \to I \times I \) (G. Peano, 1890) presents a difficulty for proofs that \( \pi_1(S^2) = \{1\} \). Explain why there is a difficulty and how some (one) method of proving this result handles this difficulty.

5. Let \( f : S^1 \times \text{int}D^2 \to S^3 \) be an embedding. Let \( \Sigma^1 = f(S^1 \times \{0\}) \) and let \( V = S^3 - \Sigma^1 \).

   (\( \Sigma^1 \) is a knot with a tubular neighborhood.) Use the Mayer-Vietoris sequence to compute \( H_1(V; \mathbb{Z}) \).

6. Use \( U(1) = S^1 \) and the fibration
   \[ U(n) \to U(n + 1) \to S^{2n+1} \]
   to compute \( \pi_1 U(n) \) for \( n \geq 1 \), \( \pi_2 U(n) \) for \( n \geq 2 \), and \( \pi_3 U(n) \) for \( n \geq 2 \).

7. In the commutative diagram of abelian groups

   \[
   \begin{array}{cccccc}
   A & \overset{i}{\to} & B & \overset{j}{\to} & C & \overset{k}{\to} & D & \overset{l}{\to} & E \\
   \downarrow{\alpha} & & \downarrow{\beta} & & \downarrow{\gamma} & & \downarrow{\delta} & & \downarrow{\epsilon} \\
   A' & \overset{i'}{\to} & B' & \overset{j'}{\to} & C' & \overset{k'}{\to} & D' & \overset{l'}{\to} & E'
   \end{array}
   \]

   the rows are exact. Show that if \( \beta \) and \( \delta \) are one-to-one, and \( \alpha \) is onto, then \( \gamma \) is one-to-one.

8. Let \( f : CP_m \to CP_m \) induce multiplication by the integer \( k \) on \( \pi_2(CP_m) \). Describe the maps \( f_* \) and \( f^* \) induced by \( f \) on the integral homology group \( H_2(CP_m; \mathbb{Z}) \) and on the integral cohomology ring \( H^*(CP_m; \mathbb{Z}) \). What is the degree of \( f \) as a map of manifolds? Cite appropriate theorems to justify your description.