Algebraic Topology Prelim  June 30, 2004

Do any five of the following eight problems. Notation: Let $T^2$ be the torus, homeomorphic to $S^1 \times S^1$.

1. Let $X$ be a path-connected space. Show that $\pi_1(X)$ is abelian if and only if for any path $\gamma$ the induced homomorphism $\pi_1(X, \gamma(1)) \to \pi_1(X, \gamma(0))$ depends only on the endpoints $\gamma(0)$ and $\gamma(1)$.

2. (a) Show that each map $f : \mathbb{R}P^2 \to T^2$ is nullhomotopic.
   (b) Show that there exists an essential map $g : T^2 \to S^2$. (It is sufficient to show that $g : \tilde{H}_*(T^2) \to \tilde{H}_*(S^2)$ is nontrivial for some map $g : T^2 \to S^2$.)

3. Place a torus inside a sphere so that the “great circle” ($S^1 \times x_0$) of the torus intersects the equator of the sphere. This produces a space $X = S^2 \cup T^2$ with the intersection $S^2 \cap T^2 = S^1$.
   (a) Find $\pi_1(X)$.  (b) Find $H_*(X)$.

4. Let $0 \to A_* \xrightarrow{\alpha} B_* \xrightarrow{\beta} C_* \to 0$ be a short exact sequence of chain complexes.
   (a) Define the connecting homomorphism $\beta : H_n(C) \to H_{n-1}(A)$.
   (b) List the choices made in the definition and show that it is independent of these choices.
   (c) Show that the sequence $H_n(B) \xrightarrow{\beta} H_n(C) \xrightarrow{\partial} H_{n-1}(A)$ is exact.

5. Let $X$ be any space and $A$ a subspace of $X$. Consider $X \cup CA$ where $CA$ is the cone $(A \times I)/(A \times 0)$ and $A \times 1$ is identified with $A \subset X$. Use excision to show that $H_*(X, A) \cong \tilde{H}_*(X \cup CA)$.

   (a) Describe the cellular structure of $M \times N$ in terms of the cellular structure of $M$ and $N$.
   (b) Possibly using (a), show that $\chi(M \times N) = \chi(M)\chi(N)$.

7. Let $f : S^2 \times S^2 \to S^2 \times S^2$ be a map which is homotopic to the map $\phi(x, y) = (-y, -x)$.
   (a) Find $f^* : H^*(S^2 \times S^2) \to H^*(S^2 \times S^2)$.
   (b) Show that $f$ must have a fixed point.

8. (a) Problem omitted due to error.
   (b) Let $M$ be a simply-connected 3-manifold. Show that $H_*(M; \mathbb{Z}_2) \cong H_*(S^3; \mathbb{Z}_2)$. 
