1) a) Define the class of *Primitive recursive* functions.
   
b) Show that if $g : \mathbb{N} \to \mathbb{N}$ is primitive recursive, and $f : \mathbb{N} \to \mathbb{N}$ by
   \[ f(n) = \sum_{m<n} g(m) \]
   then $f$ is primitive recursive.

c) Outline a proof that there is a general recursive function which is not primitive recursive.

2) a) What does it mean to say that $A \subset \mathbb{N}$ is $\Pi_n$-complete?
   
b) Let $Total = \{ e : \phi_e$ is a total function}. Show that $Total$ is $\Pi_2$-complete.

3) Let $Pr(x, y)$ be the predicate in the language of arithmetic representing the relation “$x$ codes a proof from $PA$ of the formula with Gödel code $y$” and let $Pr^*(x, y)$ represent “$x$ codes a proof from $PA$ of the negation of the formula with Gödel code $y$”. Let $\Psi(v)$ be the formula
   \[ \forall x \ Pr(x, v) \to \exists z < x Pr^*(z, y). \]
   Let $\phi$ be the Rosser sentence such that $PA \vdash \phi \iff \Psi([\phi])$.
   
a) Show that $PA \not\vdash \phi$.
   
b) Show that $PA \not\vdash \neg \phi$.

4) Let $T$ be an $L$-theory such that if $A \models T$ and $B \subset A$, then $B \models T$. Let $\Gamma = \{ \phi : \phi$ is universal and $T \models \phi \}$. Show that if $C \models \Gamma$, then $C \models T$.

5) Let $T$ be a complete theory in a countable language.
   
a) Suppose that $|S_n(T)| \leq \aleph_0$ for all $n$. Show that there is a countable $\aleph_0$-saturated model of $T$.
   
b) Sketch the proof that if $T$ is not atomic, then $|S_n(T)| = 2^{\aleph_0}$ for some $n$.

6) Let $T$ be a complete theory in a countable language. Suppose that $|S_n(T)| \leq \aleph_0$ for all $n \in T$. Let $t_0, \ldots, t_n, \ldots$ list all elements of $\bigcup S_n(T)$. Let $X \subset \omega$. Suppose for each $n$ there is $M_{X,n}$ which for $i \leq n$, $M_{X,n}$ realizes $t_i$ if and only if $i \in X$. Show that there is $M_X$ such that for all $i$, $M_X$ realizes $t_i$ if and only if $i \in X$.

7) Let $T$ be a complete theory in a countable language. Prove $A \models T$ is prime if and only if $A$ is atomic and countable.

8) a) Show that every $\aleph_0$-saturated model is $\aleph_0$-homogeneous.
   
b) Show that every atomic model is $\aleph_0$-homogeneous.