Let $\Omega = \mathbb{N}$, the set of natural numbers. Define

$$\mathcal{A} = \{ A \subset \mathbb{N} : A \text{ or } A^c \text{ is finite.} \}$$

(a) Show that $\mathcal{A}$ is a field.
(b) Is $\mathcal{A}$ a $\sigma$-field? Verify your answer.
(c) Define the set function $P$ by

$$P(E) = \begin{cases} 0, & \text{if } E \text{ is finite,} \\ 1, & \text{if } E^c \text{ is finite.} \end{cases}$$

If $A_n \in \mathcal{A}$ and $A_n \downarrow \emptyset$, can we conclude that $P(A_n) \downarrow 0$? Verify your answer.

Let $X$ be a random variable, that is, a measurable map from some probability space $(\Omega, \mathcal{B}, P)$ to the real line $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, where $\mathcal{B}(\mathbb{R})$ is the Borel subsets of $\mathbb{R}$. Define the function $F$ by

$$F(x) = P[X \leq x], \forall x \in \mathbb{R}$$

Assume $F$ is continuous.
(a) Show that $Y = F(X)$ is a random variable too.
(b) Show that $Y$ has a uniform distribution, that is,

$$P[Y \leq y] = y, \forall y \in [0, 1]$$

(c) Use Fubini’s theorem to show that $E(Y) = 1/2$.

Consider the probability space

$$(\Omega, \mathcal{B}, P) = ((0, 1], \mathcal{B}((0, 1]), \lambda)$$

where $\mathcal{B}((0, 1])$ is the Borel subsets of $(0, 1]$, and $\lambda$ is Lebesgue measure. We write $\omega \in (0, 1]$ using its dyadic expansion

$$\omega = \sum_{n=1}^{\infty} \frac{d_n(\omega)}{2^n} = .d_1(\omega)d_2(\omega)d_3(\omega)\cdots$$

where each $d_n(\omega)$ is either 0 or 1. If a number such as 1/2 has two possible expansions

$.1000\cdots$ and $.0111\cdots$,

we agree to use the nonterminating one, that is, the latter. Then each $d_n$ is a discrete random variable with possible values 0 and 1.
(a) Find $\lambda([d_n = 1])$. Verify your answer.
(b) Show that $\{d_n, n \geq 1\}$ is independent.

Suppose $\{X_n, n \geq 1\}$ are identically distributed with finite variance.
(a) Show that $nP \left[ |X_1| \geq \epsilon \sqrt{n} \right] \to 0$

(b) Show that

$$\frac{\sum_{i=1}^{n} |X_i|}{\sqrt{n}} \overset{p}{\to} 0$$

[5 ] Let $Y_s$ be Poisson distributed with parameter $s > 0$ so that

$$P[Y_s = k] = e^{-s} \frac{s^k}{k!}, \ k = 0, 1, 2, \ldots$$

(a) Show that the characteristic function of $Y_s$ is

$$\exp\{s(e^{it} - 1)\}$$

(b) Show that

$$\frac{Y_s - s}{\sqrt{s}} \Rightarrow N(0, 1), \ \text{as} \ s \to \infty$$

(Hint: The characteristic function of $N(0, 1)$ is $\exp\{-t^2/2\}$.)

[6 ] Let $\{Y_n, \ n \geq 0\}$ be iid normal random variables with mean $\mu > 0$ and variance 1. Define $X_0 = 1$ and

$$X_{n+1} = X_n + Y_n, \ n \geq 0$$

(a) Show that

$$E(e^{tX_n}) = \exp\{(1 + n\mu)t + nt^2/2\}, \ n \geq 0$$

(b) Define

$$Z_n = \exp\{-2\mu X_n\}, \ n \geq 0$$

Show that $\{Z_n, \ n \geq 0\}$ is a martingale. Specify your corresponding $\sigma$-fields.

(c) Assume that

$$E(Z_0) \geq E\left(Z_{\nu}1_{[\nu<\infty]}\right)$$

for any stopping time $\nu$. Show that

$$P\left(\bigcup_{n=1}^{\infty} [X_n < 0]\right) \leq e^{-2\mu}$$