Problem 1. Suppose $\Omega$ and $\Omega'$ are two sets. $X : \Omega \to \Omega'$ is a map with domain $\Omega$ and range $\Omega'$. Then $X$ determines a function

$$X^{-1} : \mathcal{P}(\Omega') \to \mathcal{P}(\Omega)$$

defined by:

$$X^{-1}(A') = \{\omega \in \Omega : X(\omega) \in A'\}.$$ 

Now show that if $\mathcal{C}'$ is a class of subsets of $\Omega'$ then

$$X^{-1}(\sigma(\mathcal{C}')) = \sigma(X^{-1}(\mathcal{C}'))$$

that is, the inverse image of the $\sigma$-field generated by $\mathcal{C}'$ in $\Omega'$ is the same as the $\sigma$-field generated in $\Omega$ by the inverse images.

Problem 2. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space.

(a) (Borel-Cantelli Lemma) Let $\{A_n\}$ be any events in the probability space. Prove that if

$$\sum_n \mathbb{P}(A_n) < \infty,$$

then

$$\mathbb{P}(\limsup \{A_n \text{ i.o.}\}) = \mathbb{P}(\lim_{n \to \infty} \sup A_n) = 0.$$

(b) Let $f_n, n = 1, 2, \ldots$ and $f$ be measurable functions on the probability space. Suppose for all non-negative integers $k$ and $n$ we have

$$\mathbb{P}\left\{\omega : |f_n(\omega) - f(\omega)| > \frac{1}{k}\right\} \leq 2^{-n}.$$

Show that $f_n$ converges to $f$ almost surely. (Hint: use the Borel-Cantelli Lemma.)

Problem 3. Let $(\Omega, \mathcal{B}, \mathbb{P})$ be a probability space and $X$ an $L^p(\Omega)$ random variable for some $p > 0$. Show that for any $\lambda > 0$

$$\mathbb{P}\{|X| \geq \lambda\} \leq \frac{1}{\lambda^p} \mathbb{E}|X|^p.$$
1. Suppose that $X_1, \ldots, X_n$ are i.i.d. with density
   \[ f(x|\theta) = \frac{1}{\Gamma(7)} \frac{\theta^7 |x|^6 e^{-\theta|x|}}{2}, \quad x \in \mathbb{R}, \]
   where $\theta \in (0, \infty)$. The m.l.e. of $\theta$ is easily shown to be
   \[ \hat{\theta}_n = \frac{7n}{\sum_{i=1}^n |x_i|}. \]
   i Find a MVUE of $\theta$. Prove your assertion.
   ii Among the estimators of the form $c/\sum_{i=1}^n |X_i|$ with $c$ being a constant, find the one with the minimum mean square error.
   iii Does the mean square error of any estimator of the form $c/\sum_{i=1}^n |X_i|$ achieve the Cramer-Rao lower bound for estimating $\theta$? Prove your assertion.

2. Suppose $X_1, \ldots, X_n$ are iid with pdf $f(x|\theta)$. An estimate of $\theta$ can be obtained by solving
   \[ \hat{\theta} = \arg \min_a \sum_{i=1}^n \rho(x_i - a), \]
   where $\rho(z) = \frac{1}{2} z^2 I(|z| \leq 1) + (|z| - \frac{1}{2}) I(|z| < 1)$.
   i Show that $\rho(z)$ is continuous and differentiable in $z$.
   ii Suppose $\theta_0$ is the true parameter and $\hat{\theta} \xrightarrow{P} \theta_0$, show that
   \[ \sqrt{n}(\hat{\theta} - \theta_0) = -\frac{n^{-1/2} \sum_{i=1}^n \rho(x_i - \theta_0)}{n^{-1} \sum_{i=1}^n \rho''(x_i - \theta_0)} + o_p(1). \]
   iii Based on (ii), obtain the asymptotic distribution of $\sqrt{n}(\hat{\theta} - \theta_0)$.

3. Let $X_1, \ldots, X_n$ be iid from $Uniform(\theta, \theta + 1)$, where $\theta \in \mathbb{R}$ and $n \geq 2$.
   i Find the joint distribution of $X_{(1)}$ and $X_{(n)}$.
   ii Show that a UMP test of size $\alpha$ for testing $H_0 : \theta \leq 0$ versus $H_1 : \theta > 0$ is of the form
   \[ \phi(X_{(1)}, X_{(n)}) = \begin{cases} 0 & X_{(1)} < 1 - \alpha^{1/n} \text{ and } X_{(n)} \leq 1, \\ 1 & \text{otherwise.} \end{cases} \]