Do exactly five out of the following six problems and Cross out the one that you do not wish to be considered for grading. Pay special attention to definitions and whenever you use a theorem, be specific what the relevant assumptions are and justify them, e.g., when you say by the dominated convergence theorem, then write down what the dominating function is!

1. Define what a submartingale is and show that if \((X_n, \mathcal{F}_n) : n \geq 1\) and \((Y_n, \mathcal{F}_n) : n \geq 1\) are submartingales, then so is \((\max(X_n, Y_n), \mathcal{F}_n) : n \geq 1\).

2. What does it mean to say \(X\) is a nonnegative random variable? Show that for such random variables

\[
\lim_{n \to \infty} n\mathbb{E}\left(\frac{1}{X} I\{X > n\}\right) = 0.
\]

3. Define what we mean by saying that the sequence of random variables \(\{X_n : n \geq 1\}\) is uniformly integrable. Suppose \(X_n, n \geq 1\), is Normal with expectation zero and variance \(\sigma_n^2\), find necessary and sufficient condition(s) on \(\sigma_n^2\) so that \(\{X_n : n \geq 1\}\) is uniformly integrable.

4. Show that for any sequence of random variables \(\{X_n : n \geq 1\}\);

\[
\sum_{n=1}^{\infty} \mathbb{E}\left(\frac{|X_n|}{1 + |X_n|}\right) < \infty \implies \sum_{n=1}^{\infty} X_n \text{ converges a.e.}
\]

5. What does it mean to say that the sequence \(\{X_n : n \geq 1\}\) is independent, identically distributed where their common distribution is uniform over \((0, 1)\)? Show that \(\prod_{i=1}^{n} X_i\) converges a.e.

6. Let \(X, Y, Z\) be three random variables such that \(Y\) is integrable. Define the conditional expectation \(\mathbb{E}(Y \mid X, Z)\) and show that if \((X, Y)\) and \(Z\) are independent, then \(\mathbb{E}(Y \mid X, Z) = \mathbb{E}(Y \mid X)\).
Problem 1. Let $X_1, \ldots, X_n$ be a random sample from a one-parameter exponential family
\[ \{ f_\theta \}_{\theta \in \Theta} \] of densities with respect to the Lebesgue measure on the real line $\mathbb{R}$, where
\[ f_\theta(x) = \exp(\theta x + d(\theta) + s(x))I_\theta(x), \quad x \in \mathbb{R}, \theta \in \Theta, \] and where $B$ and $\Theta$ are open intervals.

(a) Show that $E_\theta(X_i) = -d'(\theta)$.

(b) State the property of maximum likelihood estimation that justifies $\mathring{d'}(\theta) = -d'(\hat{\theta})$.

(c) Show that there exists a unique maximum likelihood $\hat{\theta}$ estimator of $\theta$.

(d) Show that $\bar{X}$ is the unique maximum likelihood estimator of $E_\theta(X_i)$.

Problem 2. Suppose that $T_1$ and $T_2$ are two uniformly minimum variance unbiased estimators of $q(\eta)$ with finite variances. Consider the unbiased estimator $T = (T_1 + T_2)/2$ and use the correlation inequality to show that $T_1 = T_2$, almost surely.

Problem 3. Let $Y$ be a Poisson random variable, $P\{Y = y\} = \lambda^y e^{-\lambda}/y!$, $y = 0, 1, 2, \ldots$, $\lambda > 0$.

(a) Determine the Fisher Information $I(\lambda), \lambda > 0$.

Let $H(Y) = 1$ if $Y = 0$, and let $H(Y) = 0$, otherwise. Calculate the Information Inequality Bound for $H(Y)$. Show that it is strictly less than the variance of $H(Y)$, uniformly in $\lambda > 0$.

Show that nevertheless, $H(Y)$ is a uniformly minimum variance unbiased estimator of its expectation.