I.
I. State precisely a version of the following theorems:
   a. Brouwer's fixed point theorem.
   b. Strong separation theorem for convex sets.
   c. Caratheodary theorem on the convex hull of sets in $\mathbb{R}^n$.
   d. Minimax theorem for matrix games.
   e. Nash equilibrium in mixed strategies.
   f. Krein- Milman theorem for closed bounded convex sets in $\mathbb{R}^n$.
   g. Completely mixed games.

II. Given an $m \times n$ matrix $A = (a_{ij})$ when can you say that the matrix admits a saddle point. Check whether $a_{ij} = j^2 - i^2$ has a saddle point.

III. Tom has two dice. One is the ordinary die with the six faces numbered 1, 2, ..6. The second one has only the numbers 1, 2, 3 where two opposite faces are numbered 1, two opposite faces are numbered 2 and two opposite faces are numbered 3. Tom picks one of the two dice secretly, tosses it once and announces the outcome to Dick. It is Dick's turn to guess the dice. Dick pays $1 to Tom for wrong guess and pays nothing for correct guess. Find a suitable payoff and solve for optimal strategies.

IV. Consider the prisoner's dilemma with payoffs

\[
\begin{pmatrix}
(3, 3) & (0, 4) \\
(4, 0) & (1, 1)
\end{pmatrix}
\]

Suppose the prisoners play the game $k$ rounds remembering all the past moves and choices by the two players. Find a Nash equilibrium strategy and payoff for the players for $k = 1, 2, 3$. Show that the Nash equilibrium strategy is unique for all $k < \infty$. Is there any deviation for the case $k = \infty$ (For this case we take as the payoff the liminf of averages over the infinite period.

V. What are the axioms of Nash on Bargaining games? Derive Nash bargaining solution using the axioms.

VI. Consider a zero-sum two person stochastic game with two states and with immediate payoffs given by

\[
A = \begin{bmatrix}
4 & 3 \\
0 & 7
\end{bmatrix}, \quad \begin{bmatrix}
1 & 6 \\
6 & 1
\end{bmatrix}
\]

The transitions depend only on the actions of player II (column player, the minimizer). In each state if column $j$ is chosen, the game moves to state $j$. Find the value of the stochastic game for discount factor $\beta = .8$. What is the undiscounted value?