Problem 1. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from a normal distribution $N(\mu, \sigma^2)$ with $\sigma^2 = 1$. Show, in two ways, to be specified below, that the uniformly minimum variance unbiased estimator of $P_{\mu}(X_1 > 0) = \Phi(\mu)$, $\mu \in \mathbb{R}$, is $\Phi\left(\frac{\bar{X}}{\sqrt{n-1}}\right)$. Explain all of your steps.

a) Derive the UMVUE. Start with the unbiased estimator $I_{(0,\infty)}(X_1)$. Then utilize the fact that the conditional distribution of $X_1$, given $\bar{X} = \xi$, is $N\left(\xi, \frac{n-1}{n}\right)$.

b) Verify the UMVUE. Find the expectation of $\Phi\left(\frac{\bar{X}}{\sqrt{n-1}}\right)$. Here, you may utilize the fact that

$$\int_{\mathbb{R}} \Phi(a + \beta z) \varphi(z) dz = P(Z_2 \leq a + \beta Z_1),$$

where $Z_1$ and $Z_2$ are i.i.d. $N(0,1)$.

Problem 2. Let $X_{ij} \sim N(\theta_i, \sigma^2)$, $\theta_i \in \mathbb{R}$, $j = 1, \ldots, n_i$, $i = 1, \ldots, k$, be independent, where $\sigma^2 > 0$ is known. Suppose we want to decide, in the Bayes approach, which of the $k$ normal populations has the largest mean $\theta_{[k]} = \max\{\theta_1, \ldots, \theta_k\}$. Assume that apriori, $\Theta_1, \ldots, \Theta_k$ are i.i.d. $N(\nu, \tau^2)$, where $\nu \in \mathbb{R}$ and $\tau^2 > 0$ are known.

a) Reduce the model by sufficiency to $X_i := (X_{i1} + \ldots + X_{in_i})/n_i$, $i = 1, \ldots, k$.

b) Determine the posterior distribution of $(\Theta_1, \ldots, \Theta_k)$, given $X_i = x_i$, $i = 1, \ldots, k$. Explain why this can be done separately for each of the $k$ populations. Then you may utilize the fact that in the special case of $k = 1$ and $n_i = 1$, the posterior distribution of $\Theta_i$, given $X_i = x_i$, is $N\left(\frac{\tau^2 x_i + \sigma^2 \nu}{\tau^2 + \sigma^2}, \frac{\tau^2 \sigma^2}{\tau^2 + \sigma^2}\right)$.

c) Determine the marginal distribution of $(X_1, \ldots, X_k)$. Again, this can be done separately for each of the $k$ populations. Then you may utilize the fact that in the special case of $k = 1$ and $n_i = 1$, the marginal distribution of $X_i$ is $N\left(\nu, \sigma^2 + \tau^2\right)$.

d) Find the Bayes selection rule under the zero-one loss function

$$L((\theta_1, \ldots, \theta_k), i) = 1 - I_{(\theta_{[k]}, i)}(i), \quad \theta_1, \ldots, \theta_k \in \mathbb{R}, i \in \{1, \ldots, k\}.$$

e) Find the Bayes selection rule under the linear loss function

$$L((\theta_1, \ldots, \theta_k), i) = \theta_{[k]} - \theta_i, \quad \theta_1, \ldots, \theta_k \in \mathbb{R}, i \in \{1, \ldots, k\}.$$
Problem 3. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from a Gamma distribution with Lebesgue density

$$f_{\alpha, \beta}(x) = \frac{1}{\Gamma(\alpha) \beta^\alpha} x^{\alpha-1} \exp\left\{-\frac{x}{\beta}\right\} I_{(0,\infty)}(x), \quad x \in \mathbb{R}, \beta > 0,$$

where the shape parameter $\alpha > 0$ is known.

a) Show that $Y := X_1 + \ldots + X_n$ is sufficient for $\beta > 0$.

b) What is the distribution of $Y$? (If you don’t know it continue with the special case of $n = 1$).

c) Switch to the parametrization $\theta := 1/\beta$ and represent the distribution of $Y$ as an exponential family in natural form.

d) Given $\alpha \in (0,1)$ and $\beta_0 > 0$, what is the uniformly most powerful level $\alpha$ test for $H_0 : \beta \leq \beta_0$ versus $H_1 : \beta > \beta_0$?

e) Given $\alpha \in (0,1)$ and $\beta_0 > 0$, what is the uniformly most powerful unbiased level $\alpha$ test for $H_0 : \beta = \beta_0$ versus $H_1 : \beta \neq \beta_0$?