Problems in Mathematical Statistics: Stat 511 and 512

Problem 1. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$. Suppose we want to test under the zero-one loss, for a fixed given $\sigma_0^2 > 0$, $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_A : \sigma^2 > \sigma_0^2$ at a given level $\alpha \in (0, 1)$. Here we look for a uniformly best invariant level $\alpha$ test.

(a) Show that the testing problem is invariant under common location shifts of $X_1, \ldots, X_n$.

(b) Show that after a reduction by sufficiency to $S = X$ and $T = \sum_{i=1}^n (X_i - X)^2$ the statistic $T$ is maximal invariant. Then explain why every invariant test must be a function of $T$.

(c) We know that at every fixed $\mu \in \mathbb{R}$ and $\sigma^2 > 0$ the distribution of $T'/\sigma^2$ is a chi-square distribution with $n - 1$ degrees of freedom, which has the Lebesgue density

$$h_{n-1}(s) = \frac{1}{2^{n/2} \Gamma((n-1)/2)} s^{(n-1)/2 - 1} \exp\left(-\frac{1}{2} s\right) I_{(0, \infty)}(s), \quad s \in \mathbb{R}.$$ 

Use this fact to show that the family of distributions of $T$ has a monotone likelihood ratio.

(d) Determine the uniformly best invariant level $\alpha$ test for $H_0$ versus $H_A$.

Problem 2. Let $X_1, \ldots, X_n$ be an i.i.d. random sample from $N(\mu, \sigma^2)$, $\mu \in \mathbb{R}$, $\sigma^2 > 0$. Suppose we want to test under the zero-one loss, for a fixed given $\sigma_0^2 > 0$, $H_0 : \sigma^2 \leq \sigma_0^2$ versus $H_A : \sigma^2 > \sigma_0^2$ at a given level $\alpha \in (0, 1)$. Here we look for a uniformly best unbiased level $\alpha$ test.

(a) Represent the joint distribution of $X_1, \ldots, X_n$ as an exponential family with natural parameter $\vartheta = (\tau, \xi) := (-1/(2\sigma^2), \mu/\sigma^2)$ and generating statistic $(U(x), V(x)) = (\sum_{i=1}^n x_i^2, \sum_{i=1}^n x_i)$. You may do that first for $n = 1$, and then adjust the result with some simple arguments to $n \geq 1$.

(b) Set up the uniformly best unbiased level $\alpha$ test for $H_0$ versus $H_A$ as a conditional level $\alpha$ test based on $U(x)$, given $V(x)$. Show that at every $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ it can be written as

$$\varphi(U(x), V(X)) = I_{[c(\alpha, V(x)), \infty)}(U(x) - n^{-1}V(x)^2),$$

where $c(\alpha, v)$ is determined by $E_{\sigma_0^2}(\varphi(U(X), V(X))|V(X) = v) = \alpha$, $v \in \mathbb{R}$.

(c) Argue that by the independence of $W(X) := U(X) - n^{-1}V(X)^2$ and $V(X)$ the quantity $c(\alpha, v)$ does not depend on $v \in \mathbb{R}$. Thus the test is an unconditional level $\alpha$ test, and in fact the same as in Problem 1(d).
Problem 3. Let \( X_1, \ldots, X_n \) be an \( i.i.d. \) random sample from a uniform distribution \( U(0, \theta) \), \( \theta \in \Delta = (0, \infty) \). Suppose we want to estimate \( \theta \) under the squared error loss. Utilize without proof that \( Y = \max\{X_1, \ldots, X_n\} \) is a sufficient statistic and that it has the Lebesgue density

\[
f_\theta(y) = \frac{n}{\theta^n} y^{n-1} I_{[0, \theta]}(y), \quad y \in \mathbb{R}.
\]

Assume that a priori \( 1/\Theta \) follows a gamma distribution \( \text{Ga}(\lambda, \beta) \) for some fixed \( \lambda, \beta > 0 \), which has the Lebesgue density

\[
\text{ga}_{\lambda, \beta}(\theta) = \frac{\beta^\lambda}{\Gamma(\lambda)} \theta^{\lambda-1} \exp\{-\beta \theta\} I_{(0, \infty)}(\theta), \quad \theta \in \mathbb{R}.
\]

(a) Determine the Lebesgue density of the prior distribution of \( \Theta \), which is called the inverse gamma distribution.

(b) Determine the Bayes estimator of \( \theta \in \Delta \). Note that this does not lead to a simple form, but still includes integrals.