1. [Stat521, 20 points]

Let $A$ and $B$ be $n \times n$ symmetric matrices. Suppose a random vector $Y \sim N_n(\mu, \sigma^2 I_n)$, where $\mu = (\mu_1, \ldots, \mu_n)'$.

(i) If $A^2 = A$, what is the distribution of $Y'AY$?
(ii) Show that $Y'AY$ and $Y'BY$ are independent if $AB = 0$.

2. [Stat521, 20 points]

(i) Let $T$ be an MVUE of a single parameter $\theta$. Show that $Cov(T, D) = 0$ for any estimator $D$ with mean 0.
(ii) Let $T_1, \ldots, T_k$ are MVUE’s of the parameters $\theta_1, \ldots, \theta_k$, respectively. Show that $c_1 T_1 + \cdots + c_k T_k$ is the MVUE of $c_1 \theta_1 + \cdots + c_k \theta_k$.

3. [Stat521, 20 points]

Consider a random vector $Y = (y_1, \ldots, y_n)'$ satisfying

$$Y = X\beta + \varepsilon,$$

where $X$ is an $n \times m$ known matrix with rank $r \leq m \leq n$, $\beta = (\beta_1, \ldots, \beta_m)'$ is a vector of unknown parameters, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)'$ is a random vector with $E(\varepsilon) = 0$ and $Var(\varepsilon) = \sigma^2 I_n$ with an unknown parameter $\sigma^2 > 0$.

(i) Derive the least square estimator $\hat{\beta}$ of $\beta$. When is $\hat{\beta}$ unique?
(ii) Let $P$ be an $m \times 1$ nonzero vector. Show that $P'\hat{\beta}$ is unique for all solutions $\hat{\beta}$ if and only if $P \in \mathcal{M}(X')$, where $\mathcal{M}(X')$ represents the column space of $X'$. 
4. [Stat522, 20 points]
Supposed that $Z_1, \ldots, Z_n$ are i.i.d. $N_p(0, \Sigma)$. Let $A = \sum_{i=1}^n Z_i Z_i^T$, then it is known that $A$ follows Wishart distribution $W_p(n, \Sigma)$. Partition $A$ and $\Sigma$ into $q$ and $p - q$ rows and columns,

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix}.$$

(i) Show that $A_{11} \sim W_q(n, \Sigma_{11})$.
(ii) Denote $\Sigma_{11.2} = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$ and $A_{11.2} = A_{11} - A_{12} A_{22}^{-1} A_{21}$. Show that $A_{11.2} \sim W_q(n - p + q, \Sigma_{11.2})$, and $A_{11.2}$ is independent of $A_{22}$.

5. [Stat522, 20 points]
Given a multivariate regression model with response $Y$ and design matrix $X$

$$Y_{n \times m} = X_{n \times (p+1)} \beta_{(p+1) \times m} + \varepsilon_{n \times m},$$

where $\varepsilon_{n \times m} = (\varepsilon'_1, \ldots, \varepsilon'_n)$ with $\varepsilon_i \overset{i.i.d.}{\sim} N_m(0, \Sigma)$.

(i) Show that the m.l.e. of $\beta$ is the same as the solution of

$$\min_{\beta} \text{tr} \left( (Y - X\beta)'(Y - X\beta) \right),$$

regardless of the covariance matrix $\Sigma$.
(ii) Partition $\beta = \begin{pmatrix} \beta_{(1)} \\ \beta_{(2)} \end{pmatrix}$ with $r + 1$ and $p - r$ rows, respectively. Conduct exact likelihood ratio test for $H_0 : \beta_{(2)} = 0$ against $H_a : \beta_{(2)} \neq 0$. Please give as much details as possible.

6. [Stat522, 20 points]
Supposed that $(x_1, y_1), \ldots, (x_n, y_n)$ are i.i.d. from $X|Y = y \sim N_p(\mu_y, \Sigma)$ with $y \in \{1, 2\}$. Consider Fisher’s linear discriminant analysis (LDA) with $\pi_y = P(Y = y)$ for $y = 1, 2$.

(i) Show that the LDA rule classifies to class 2 if $x' \hat{\Sigma}^{-1}(\hat{\mu}_2 - \hat{\mu}_1) \geq c$, where $c$ is some known constant.
(ii) Consider the least squared error formulation, \( \min(\beta_0, \beta) \sum_{i=1}^{n} (y_i - \beta_0 - \beta'x_i)^2 \). Show that the solution \( \hat{\beta} \) satisfies that \( \hat{\beta} \propto \hat{\Sigma}^{-1}(\hat{\mu}_2 - \hat{\mu}_1) \).

(iii) Can you give another reasonable loss function as opposed to the square error loss in (ii)? Compare it against the formulation in (ii), and discuss their advantages and disadvantages. Please give as much details as possible.