Math 210 Final Exam Review Problems

Show **all** of your work. An unjustified answer is not correct!
Write neatly and clearly, and indicate your answer to each problem.

Line Integrals and Vector Fields

Problem 1: Calculate $\int_C y\ dx + (x+z)\ dy + y\ dz$ along the curve C given by $\vec{r}(t) = \langle t, t^2, t^3 \rangle$ for $0 \leq t \leq 1$.

Problem 2: Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y,z) = \langle yz, xz, xy \rangle$ and C is the given by $\vec{r}(t) = \langle t, 2t, t^2 \rangle$ for $-1 \leq t \leq 1$.

Problem 3: Calculate $\int_C x\ dx + y\ dy$ where C is the line segment from $(0,0)$ to $(1,2)$.

Problem 4: Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle x^2 - y^2, 2xy \rangle$ and C is the line segment from $(0,0)$ to $(1,1)$.

Problem 5: Calculate $\int_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle ye^{xy}, xe^{xy} \rangle$ and C is the arc of the ellipse $x^2 + 4y^2 = 16$ from $(0,2)$ to $(-4,0)$.

Green's Theorem

Problem 6: Calculate $\oint_C \vec{F} \cdot d\vec{r}$ where $\vec{F}(x,y) = \langle 2y + e^x, x + \sin(y^2) \rangle$ and C is the curve given by $\vec{r}(t) = \langle 3\cos(2t), 3\sin(2t) \rangle$ for $0 \leq t \leq \pi$.

Problem 7: Calculate $\oint_C xy\ dx + x^2 e^y dy$ where C is the boundary of the square with vertices $(3,0), (3,2), (1,2), (1,0)$.

Problem 8: Calculate $\oint_C xy\ dx + y^5 dy$ where C is the boundary of the triangle with vertices $(0,0), (2,0), (2,1)$.

Problem 9: Calculate $\oint_C x^2 y\ dx + xy^5 dy$ where C is the boundary of the square with vertices $(1,1), (1,-1), (-1,-1), (-1,1)$.

Problem 10: Calculate $\oint_C x^2 y\ dx$ where C is the boundary of the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 9$.

Conservative Vector Fields

Determine whether or not each vector field $\vec{F}$ is conservative.
If it is, find a function f with $\nabla f = \vec{F}$.

Problem 11: $\vec{F}(x, y) = \langle 3x^2 - 4y, 4y^2 - 2x \rangle$.

Problem 12: $\vec{F}(x, y) = \langle x + y^2, 2xy + y^2 \rangle$.

Problem 13: $\vec{F}(x, y) = \langle x^2 - y^2, 2xy \rangle$.

Problem 14: $\vec{F}(x, y) = \langle e^{2x} + x\sin(y), x^2\cos(y) \rangle$.

Problem 15: $\vec{F}(x, y) = \langle ye^x + \sin(y), e^x + x\cos(y) \rangle$.

Problem 16: $\vec{F}(x, y) = \langle x^2 + y, x^2 \rangle$.

Problem 17: $\vec{F}(x, y) = \langle 3x^2y^4, 1 + 4x^3y^3 \rangle$.

Problem 18: $\vec{F}(x, y) = \langle xy, x^2 \rangle$.

Problem 19: $\vec{F}(x, y) = \langle y, -x \rangle$.

Problem 20: $\vec{F}(x, y) = \langle y, x \rangle$.

Problem 21: Consider the vector field $\mathbf{F}(x, y) = \langle 1 + 2ye^{2x}, 2y + e^{2x} \rangle$

a) Show that $\mathbf{F}$ is conservative.

b) Find a function f such that $\mathbf{F} = \nabla f$.

c) Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ along the curve C given by $\mathbf{r}(t) = \langle 1 - t, te^t \rangle$ for $0 \leq t \leq 1$.

Problem 22: One of the two vector fields below is the gradient of a function $h(x, y)$, and the other cannot be the gradient of a function.

$$\mathbf{F} = (x^2 + y)\vec{i} + x^2\vec{j} \qquad \mathbf{G} = 3x^2y^4\vec{i} + (1 + 4x^3y^3)\vec{j}$$

(a) Use partial derivatives to determine which of the fields cannot be a gradient.
(b) Find a function whose gradient is the other field.

Double Integrals

For each of these problems, it is strongly recommended that you draw a careful plot of the region of integration, and label the coordinates of important points of intersection.

Problem 23: Find the volume inside the cylinder $x^2 + y^2 = 4$, above the plane $z = 0$ and below the plane $x + 2y + z = 6$.

Problem 24: Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below the plane $z = 0$, and inside the cylinder $x^2 + y^2 = 9$.

Problem 25: Find the volume of the bounded region lying above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 9$.

Problem 26: Find the volume of the region bounded by the two paraboloids $z = x^2 + y^2$ and $z = 18 - x^2 - y^2$.

Problem 27: Change to polar coordinates and evaluate $\iint_{D_R} e^{-x^2-y^2} dA$ where $D_R = \{(x, y) \colon x^2 + y^2 \leq R^2\}$.

Problem 28: Find the mass of a lamina of density $\rho(x, y) = \sqrt{x^2 + y^2}$ contained in the upper half plane, and bounded by the circle $x^2 + y^2 = 4$.

Problem 29: Calculate the total mass of the region bounded by the graphs of $z = x^2+y^2-8$ and $z = 10 - x^2 - y^2$ with density $\rho(x, y, z) = x^2 + y^2$.

Problem 30: Change the order of integration: $\int_0^1 \int_{x^2}^1 f(x, y)\ dy\ dx$

Problem 31: Change the order of integration: $\int_0^1 \int_{2x^2}^{2x} f(x, y)\ dy\ dx$

Problem 32: Change the order of integration and evaluate: $\int_0^1 \int_y^1 e^{-x^2}\ dx\ dy$

Problem 33: Change the order of integration and evaluate: $\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\ dx\ dy$

Surface Area

Problem 34: Find the surface area of the part of the plane $3x + 4y + 5z = 20$ lies above the rectangle $[0, 4] \times [-2, 3]$.

Problem 35: Find the area of the part of the surface $z = x^2 + y^2$ over the disc $x^2 + y^2 \leq 9$.

Problem 36: Find the surface area of the part of the paraboloid $z = 1 - x^2 - y^2$ which lies above the plane $z = -3$.

Problem 37: Find the surface area of the part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.

Change of Variables

Problem 38: Use the change of variables $x = 2u + 3v$, $y = 3u - 2v$ to calculate the integral

$$\iint_R (x - y)\ dx\, dy$$

where R is the region bounded by the lines

$$2y = 3x, \quad 2y = 3x - 13, \quad 3y = -2x + 13, \quad 3y = -2x$$

Problem 39: Use the change of variables $x = (u + v)/3$, $y = (v - 2u)/3$ to calculate the integral

$$\iint_R 3x + 4y\ dx\, dy$$

where R is the region bounded by the lines

$$y = x, \quad y = x - 2, \quad y = -2x, \quad y = 3 - 2x$$

Problem 40: Use the transformation: $x = (u+v)/2$, $y = (v-u)/2$ to evaluate the integral

$$\iint_R (x - y)(x + y) dx\, dy$$

where R is the region bounded by the lines

$$y = x + 2, \quad y = x, \quad y = -x + 2, \quad y = -x + 4$$

Problem 41: a) Calculate the Jacobian of the transformation T given by:

$$x = u^2 - v^2; \quad y = 2u\, v$$

b) Using (a), evaluate the integral $\iint_R x\ dA$, where R is the region bounded by the parabolas

$$y^2 = 4 - 4x, \quad y^2 = 4 + 4x, \quad and y \geq 0$$

(Hint: R is the image of $\{(u, v) \mid 0 \leq u \leq 1, 0 \leq v \leq 1\}$ under the transformation T.)

Triple Integrals

Problem 42: Evaluate the triple integral

$$\int_0^4 \int_0^y \int_0^{\sqrt{4y^2-x^2}} z \; dz\, dx\, dy$$

Problem 43: Evaluate the triple integral $\iiint_E x \; dV$ where E is the region bounded by the planes $x = 0$, $y = 0$, $z = 0$, $3x + 2y + z = 6$.

Problem 44: Evaluate the triple integral $\iiint_E x \; dV$ where E is the region bounded by the paraboloid $x = 4y^2 + 4z^2$ and the plane $x = 4$.

Problem 45: Calculate the center of mass of the **tetrahedron** bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $2x + y + z = 6$. The density is $\rho(x, y, z) = 6 - x$.

Problem 46: Evaluate $\iiint_B x^2 + y^2 + z^2 \; dx\, dy\, dz$ where B is the ball of radius $R = 3$.

Problem 47: Evaluate $\iiint_B x^2 + y^2 \; dx\, dy\, dz$ where B is the ball of radius $R = 1$.

Problem 48: Evaluate $\iiint_D z \; e^{-x^2-y^2} \; dx\, dy\, dz$ where D is the region inside the cylinder $x^2 + y^2 = 4$ bounded by the planes $z = -1$ and $z = 1$.

Problem 49: Evaluate $\iiint_D \frac{dV}{x^2 + y^2 + z^2}$ where D is the region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 4$.

Problem 50: Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$ and above the cone $z = \sqrt{x^2 + y^2}$.

Problem 51: Find the volume of the solid that lies within the sphere $x^2 + y^2 + z^2 = 4$, above the xy-plane, and below the cone $z = \sqrt{x^2 + y^2}$.

Lagrangian Multipliers

Problem 52: Use the Lagrangian multiplier method to find the maximum and minimum value of the function

$$f(x, y) = 2x^2 + 3y^2 + x$$

constrained to the unit circle $x^2 + y^2 = 1$

Problem 53: Find the maximum and minimum value of $f = x^2 - y^2 + 2y$ under the constraint $x^2 + y^2 = 1$.

Problem 54: Use the method of Lagrange multipliers to find the maximum and minimum value of the function

$$f(x, y, z) = x + 3y - z$$

subject to the constraint $x^2 + 4y^2 + z^2 = 17$.

Problem 55: Find the maximum value of the function $f(x, y, z) = x - y + 3z$ on the ellipsoid $x^2 + y^2 + 4z^2 = 4$.

Problem 56: Use the method of Lagrange multipliers to find the point on the plane $2x - y + 5z = 11$ closest to the point $(0, 4, -3)$.

Critical Points

For all the following problems, justify your answers using the second derivative tests.

Problem 57: Find the critical points of the function $f(x, y) = x^3 - 3xy + y^3$ and determine which are maxima, minima or saddles.

Problem 58: Find all of the critical points of the function $f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy$ and use the second derivative test to classify them as local maxima, local minima, or saddles, or state that the second derivative test fails.

Problem 59: Find the critical points of the function $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ and determine which are maxima, minima or saddles.

Problem 60: Find the critical points of the function $f(x, y) = x^2 + y^2 + x^2y + 1$ and determine which are maxima, minima or saddles.

Problem 61: Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the disc $x^2 + y^2 \leq 16$. (That is, find all interior critical points, and all max/min points on the boundary circle.)

Chain Rule

Problem 62: Suppose $z = x\,y^2 - x\,\sin y$, and $x = t^2$ and $y = 2\,t^3$. Use the chain rule to calculate $\dfrac{dz}{dt}$. Simplify your answer.

Problem 63: Suppose $z = x\,\ln(x + 2y)$, and $x = \sin t$ and $y = \cos t$. Use the chain rule to calculate $\dfrac{dz}{dt}$. Simplify your answer.

Problem 64: Suppose $w = 3x + x\,z + y^2$, and $x = t^2$, $y = t$, $z = 5t$. Use the chain rule to calculate $\dfrac{dw}{dt}$. Simplify your answer.

Problem 65: Suppose $z = x^2 + x\,y + y^2$, and $x = s + t$, $y = s\,t$. Use the chain rule to calculate $\dfrac{dw}{dt}$. Simplify your answer.

Problem 66: Suppose $z = \dfrac{x}{y}$, and $x = s\,e^t$, $y = 1 + s\,e^{-t}$. Use the chain rule to calculate $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$. Simplify your answer.

Problem 67: Suppose that $f(x, y)$ is a differentiable function and that $h(s, t)$ is the composite function $h(s, t) = f(s^2 + t^2, s^2t^2)$. Suppose that the first partial derivatives of f satisfy

$$\frac{\partial f}{\partial x}(13, 36) = \frac{\partial f}{\partial y}(13, 36) = 3.$$

Compute $\frac{\partial h}{\partial s}(2, 3)$ and $\frac{\partial h}{\partial t}(2, 3)$.

Tangent Planes

Problem 68: Find an equation for the tangent plane to the graph of the function $f(x, y) = xy - x^3$ at the point where $(x, y) = (2, 5)$.

Problem 69: Find an equation of the tangent plane to the graph of the function $f(x.y) = e^x \ln y$ at the point $(3, 1, 0)$.

Problem 70: Find an equation for the tangent plane to the graph of the function $f(x, y) = \ln(2x + y)$ at the point $(-1, 3, 0)$.

Problem 71: Find the equation of the tangent plane at $(1, 2, -1)$ to the level surface

$$x^2 + xyz + 2z^3 + 3 = 0$$

Directional Derivatives

Problem 72: Find the directional derivative $D_v f(1, \pi, 0)$ where $f(x, y, z) = x^2 + \sin(y) + e^z$ and $\vec{v} = \langle 1, 0, 1 \rangle$.

Problem 73: Find the directional derivative $D_v f(\pi, 1)$ where $f(x, y) = e^{x+y} \sin(xy)$ and $\vec{v} = \langle 4, 0 \rangle$.

Problem 74: Find the rate of increase of $f(x, y) = x^2 - x/y$ at $P = (2, -1)$ *in the direction of* $\vec{v} = \langle 3, 4 \rangle$.

Problem 75: Find the rate of increase of $f(x, y) = x\sqrt{y}$ at $P = (4, 4)$ *in the direction of* $\vec{v} = \langle 3, 1 \rangle$.

Problem 76: Find the directional derivative of $f(x, y) = x\sqrt{y}$ at $P = (4, 4)$ i*n the direction of maximal increase.*

Problem 77: a) Compute the gradient at $(x_0, y_0, z_0) = (2, 3, 1)$ of

$$f(x, y, z) = \frac{1}{x^2 + 2y^2 + 3z^2}$$

b) What direction from $(2, 3, 1)$ should we go in order to make $f(x, y, z)$ *decrease* fastest?

Problem 78: Let $f(x, y) = y \sin(xy^2)$

a) Compute the directional derivative of f at $(0, 1)$ in the direction of $(4, 3)$.

b) In what direction does f increase most rapidly at the point $(0, 1)$?

Lengths and Angles

Problem 79: A triangle has vertices at the points

$$A = (1, 1, 1), \quad B = (1, -3, 4), \quad \text{and} \quad C = (2, -1, 3).$$

(a) Find the cosine of the angle between the vectors $\vec{AB}$ and $\vec{AC}$.

(b) Find an equation of the plane containing the triangle.

(c) Find the area of the triangle.

Problem 80: A triangle has vertices at the three points

$$A = (1, 2, -3), \quad B = (-1, 0, 2), \quad \text{and} \quad C = (2, 1, -1).$$

Find the area of the triangle.

Domain and Limits

Problem 81: a) What is the domain of the function

$$f(x,y) = \frac{xy+1}{x^2+y^2+1}$$

b) Find $\lim f(x,y)$ as (x,y) approaches $(0,0)$ if it exists; or show the limit does not exist.

Problem 82: Does the function

$$f(x,y) = \frac{x^2 y}{(x^2+y^2)}$$

have a limit as $(x,y) \to (0,0)$? Show your reasoning.

Problem 83: Find the limit along the linear paths given by $x = ay$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^3}{x^2+y^2}$$

Is the limit path dependent? Explain your answer.

b) Show that

$$\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)+1}{(x^2+y^2)+4}$$

does exist for all paths. Explain why your method is path independent.

Curves

Problem 84: For the space curve given by $\vec{c}\,(t) = \langle e^t, \sin t, \cos t\rangle$.
a) Find the velocity vector $\vec{v}\,(t)$ and acceleration vector $\vec{a}\,(t)$.

b) Find a unit vector $\vec{u}$ perpendicular to the velocity and acceleration vectors at $t = 0$.

c) Find the curvature at $t = 0$.

Problem 85: Find the arclength of the curve $\vec{c}\,(t) = \langle t, t\sin t, t\cos t\rangle, \quad 0 \le t \le \pi$.

Problem 86: For the curve $\vec{r}(t) = \langle -6\cos t, 6\sin t, 8t\rangle$

a) Find the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$.

b) Find the curvature $\kappa(t)$.

Problem 87: Find the curvature of $\mathbf{r}(\mathbf{t}) = \langle e^t\cos t, e^t\sin t, t\rangle$ at the point $(1,0,0)$.

Problem 88: A curve is defined by $\mathbf{r}(t) = \langle t^3, t^2+t, 3-2t\rangle$.
a) Find the velocity $\vec{v}(t)$ and the acceleration $\vec{a}(t)$.

b) Find the curvature $\kappa(t)$.

Cross Products and Planes

Problem 89: Find two unit vectors orthogonal to the vectors $\langle 1, -1, 1\rangle$ and $\langle 0, 4, 4\rangle$.

Problem 90: a) Find a vector $\vec{c}$ perpendicular to the vectors $\vec{a} = \langle 1, 0, 6\rangle$ and $\vec{b} = \langle 2, 3, -8\rangle$.
b) What is the volume of the parallelepiped determined by the vectors $\vec{a}$, $\vec{b}$ $\vec{c}$.

Problem 91: a) Find an equation for the plane **P** containing the point $(6, 3, 2)$ and perpendicular to the vector $\langle -2, 1, 5\rangle$.
b) What is the distance from the point $(-2, 1, 5)$ to the plane **P** ?

Problem 92: Find the equation of the plane passing through the point $(1, -2, 3)$ and *orthogonal* to the planes $3x + 2y - 4z = 10$ and $x - z = 1$.

Problem 93: The three points $A = (1, 1, 0)$, $B = (0, 4, 0)$ and $C = (2, 0, 1)$ are in a plane. Starting with the vector form for the equation a plane, find the equation of the plane.

Problem 94: Find an equation of the plane passing through the points $(3, 2, 1)$, $(2, 1, -1)$ and $(-1, 3, 2)$.

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