THEORETICAL SKIN-FRICTION LAW IN A TURBULENT BOUNDARY LAYER

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ABSTRACT. We study transitional and turbulent boundary layers using a turbulent velocity profile equation recently derived from the Navier-Stokes-alpha and Leray-alpha models. From this equation we obtain a theoretical prediction of the skin-friction coefficient in a wide range of Reynolds numbers based on momentum thickness, and deduce the maximal value of $c_f^{\max} = 0.0063$ for turbulent velocity profiles. A two-parameter family of solutions to the equation matches experimental data in the transitional boundary layers with different free-stream turbulence intensity, while one-parameter family of solutions, obtained using our skin-friction coefficient law, matches experimental data in the turbulent boundary layer for moderately large Reynolds numbers.

1. INTRODUCTION

The mathematical description of turbulence, particularly boundary-layer turbulence, has been a subject of serious investigation for over a century, and still poses a great challenge for scientists. Even though the velocity field in a turbulent flow is seemingly chaotic, the averaged velocity displays some regular structure. Finding a suitable closed approximation to the Reynolds equations is one of the main problems in turbulence theory.

The Navier-Stokes-α (NS-α) model, also known as the viscous Camassa-Holm equation or LANS-α (Lagrangian averaged Navier-Stokes-alpha) model, was introduced by S. Chen, C. Foias, D. D. Holm, E. Olson, E. S. Titi, and S. Wynne in 1998. The NS-α model was proposed as a closure of the Reynolds equations, and its solutions were compared with empirical data for turbulent flows in channels and pipes. Since then, the model received significant attention from a number of scientists, resulting in numerous publications. Inspired by the NS-α equations, we introduced a similar model for 3-D turbulence, the Leray-α model (see [10]).

However, both models have a higher order than the Navier-Stokes equations, and they require additional boundary conditions. The physical boundary conditions for the models are still not known, despite continuous effort by many investigators. Because of this, computations with the models are usually performed in periodic domains, and the models are not yet used for real engineering applications.

Developing a boundary-layer theory for alpha-models is the first step in approaching this issue. We started studying a zero-pressure gradient case in [7, 8, 9], showing that the unknown boundary conditions can be uniquely determined by the physical parameters of the flow near the boundary. In this note we show that the proposed turbulent boundary-layer model predicts the skin friction for a wide range of Reynolds numbers.

1.1. Navier-Stokes-α and Leray-α models.

The Navier-Stokes-alpha model of fluid turbulence, introduced in [3, 4, 5] is written as

$$(1)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} + v_j \nabla u_j = \nu\Delta\mathbf{v} - \nabla q + f \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{v} = \mathbf{u} - \dfrac{\partial}{\partial x_i}\left(\alpha^2 \delta_{ij}\dfrac{\partial}{\partial x_j}\mathbf{u}\right), \end{cases}$$

where $\mathbf{u}$ represents the averaged physical velocity of the flow, q is a pressure analog, f is a force, and $\nu > 0$ is the viscosity. See also [14, 15, 16, 19, 20] and references therein for some results on this model.

Inspired by the Navier-Stokes-α equations, the Leray-α model was introduced in [10]. This model is given by the Leray regularization of the Navier-Stokes equation with a special smoothing filter, the inverse of the Helmholtz operator:

$$(2)\qquad \begin{cases} \dfrac{\partial}{\partial t}\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{v} = \nu\Delta\mathbf{v} - \nabla p + f \\ \nabla\cdot\mathbf{u} = 0 \\ \mathbf{v} = \mathbf{u} - \alpha^2\Delta\mathbf{u}, \end{cases}$$

where $\mathbf{u}$ represents the averaged physical velocity of the flow, p is the averaged pressure, f is the force, and $\nu > 0$

is the viscosity. Estimates for the dimension of the global attractor, established in [10], are smaller than that of the NS-α model (see [16]), which suggests that the Leray-α model might be might easier to compute with.

In the study of a turbulent-boundary layer flow, the filter length scale α, which represents the averaged size of the Lagrangian fluctuations (see [3, 4, 5]), is considered as a parameter of the flow, changing along the streamlines in the boundary layer. More precisely, we use an assumption, that α is proportional to the thickness of the boundary layer.

1.2. **Boundary-layer approximation of the Leray-α model.** Consider a two-dimensional turbulent flow near a surface. Let x be the coordinate along the surface, y the coordinate normal to the surface, and $\mathbf{u} = (u, v)$ the velocity of the flow. Assuming that α does not depend on y, normalizing variables, and neglecting the terms of the Leray-α model that are small near the boundary (see [9, 10]), we obtain the following generalization of the Prandtl equations, the boundary-layer approximation of the 2D Leray-α model:

$$(3)\qquad \begin{cases} u\dfrac{\partial}{\partial x}w + v\dfrac{\partial}{\partial y}w = \dfrac{\partial^2}{\partial y^2}w - \dfrac{\partial}{\partial x}p, \\ \dfrac{\partial}{\partial y}p = 0, \\ \dfrac{\partial}{\partial x}u + \dfrac{\partial}{\partial y}v = 0, \end{cases}$$

where (u, v) represents the averaged velocity, p the averaged pressure, and $w = \left(1 - \alpha^2 \frac{\partial^2}{\partial y^2}\right) u$. The physical (non-slip) boundary conditions are $u(x, 0) = v(x, 0) = 0$, and $(u(x, y), v(x, y)) \to (U(x), 0)$ as $y \to \infty$. Assuming Bernoulli's equation $U^2(x) + 2p(x) = \text{const.}$ for the external velocity $U(x)$, we deduce that $\frac{\partial}{\partial x}p(x, y) = U(x)U'(x)$. Rewriting equations (3) in terms of U, we obtain

$$(4)\qquad \begin{cases} u\dfrac{\partial}{\partial x}w + v\dfrac{\partial}{\partial y}w = \dfrac{\partial^2}{\partial y^2}w + UU' \\ v(x, y) = -\displaystyle\int_0^y \frac{\partial}{\partial x}u(x, z)\, dz \\ w = u - \alpha^2 \dfrac{\partial^2}{\partial y^2}u \end{cases}$$

Our goal is to study special, in particular, self-similar solutions to (4). More precisely, we will study solutions to this system of the form

$$u(x, y) = U(x)h'(\xi, \lambda(x)),\ \alpha(x) = \beta\delta(x),\ \xi = \frac{y}{\delta(x)},$$

the functions $\lambda(x)$ and $\delta(x)$ being defined later. Here prime represents a partial derivative with respect to the first variable. For simplicity we assume $h(0) = 0$. The boundary conditions imply $h'(0, \lambda) = 0$ and $h'(\xi, \lambda) \to 1$ as $\xi \to \infty$. Then (4) reduces to

$$\frac{U}{\delta^2}\left[m''' + \delta\frac{d(\delta U)}{dx}hm'' + \delta^2\frac{dU}{dx}(1 - h'm')\right] - U^2\frac{d\lambda}{dx}\left(h'\frac{\partial m'}{\partial\lambda} - \frac{\partial h}{\partial\lambda}m''\right) = 0,$$

where $m = h - \beta^2 h''$. Take

$$\begin{aligned} \delta(x) &:= \frac{1}{U(x)}\left(\int_0^x U(z)\, dz\right)^{\frac{1}{2}}, \\ \lambda(x) &:= \frac{U'(x)}{U^2(x)}\int_0^x U(z)\, dz. \end{aligned}$$

In order for (u, v, α) to satisfy (4), it is sufficient for (h, λ) to satisfy the following equations:

$$(5)\qquad \begin{cases} \dfrac{d\lambda}{dx}\left(h'\dfrac{\partial m'}{\partial\lambda} - \dfrac{\partial h}{\partial\lambda}m''\right) = 0, \\ m''' + \frac{1}{2}hm'' + \lambda(x)(1 - h'm') = 0, \\ m = h - \beta^2 h''. \end{cases}$$

The physical boundary conditions are $h(0) = h'(0) = 0$ and $h'(\xi) \to 1$ as $\xi \to \infty$. In the zero pressure gradient case, when the exterior velocity U is constant, we have $\lambda = 0$ and (5) reduces to the generalization of the Blasius equation. Note that this equation can be also derived from the NS-α model (see [7, 8, 9]).

1.3. **Generalization of the Blasius equation.** A generalization of the Blasius equation is given by the following fifth-order ordinary differential equation:

$$(6)\qquad m''' + \tfrac{1}{2}hm'' = 0,$$

where $m = h - \beta^2 h''$. The boundary conditions are $h(0) = h'(0) = 0$, and $h'(\xi) \to 1$ as $\xi \to \infty$.

This equation describes horizontal velocity profiles $\{h'(\cdot)\}$ in transitional and turbulent boundary layers with zero pressure gradients. More precisely, for a fixed horizontal coordinate, we model the averaged velocity profiles (u, v) in the following way:

$$(7)\qquad u(y) = u_{\mathrm{e}}h'\left(\frac{y}{\sqrt{l_{\mathrm{e}}l}}\right),$$

$$(8)\qquad v(y) = \frac{u_{\mathrm{e}}}{\sqrt{R_l}}h'\left(\frac{y}{\sqrt{l_{\mathrm{e}}l}}\right).$$

Here y is the vertical coordinate, u_{e} is the horizontal velocity of the external flow, h is a solution to (6), l is a local length scale, a parameter of the flow that has to be determined, $R_l = u_\mathrm{e} l/\nu$, and l_e is the external length scale $l_\mathrm{e} = \nu/u_\mathrm{e}$.

The dimensionless parameter β, a filter width scale defining a turbulent flow regime, represents the ratio of the averaged size of turbulent fluctuations to the boundary layer thickness.

Observe that when β is zero, i.e., when there are no turbulent fluctuations, we have $m = h$ and equation (6) reduces to the Blasius equation. The Blasius equation has a unique solution h_B satisfying physical boundary conditions (see [24]). The Blasius profile h'_B has one inflection point in logarithmic coordinates (see [8]), and it matches experimental data in the laminar region of the boundary layer.

It is easy to obtain a skin-friction law for the Blasius profile. Indeed, the skin-friction coefficient for the Blasius profile is

$$c_f := \frac{2\nu}{u_e^2}\frac{\partial u}{\partial y}\Big|_{y=0} = \frac{2}{\sqrt{R_l}} h''_\mathrm{B}(0),$$

where l is the distance from the edge of the plate. On the other hand, the Reynolds number based on momentum thickness is given by

$$\begin{aligned} R_\theta &= \frac{1}{\nu}\int_0^\infty u(y)\,(1 - u(y)/u_\mathrm{e})\;dy \\ &= \sqrt{R_l}\int_0^\infty h'_\mathrm{B}(\xi)(1 - h'_\mathrm{B}(\xi))d\xi \\ &= c_\mathrm{B}\sqrt{R_l}, \end{aligned}$$

where $c_B \approx 0.664$. Therefore, the skin-friction coefficient satisfies the following law in the laminar boundary layer:

$$c_f = \frac{2h''_\mathrm{B}(0)c_\mathrm{B}}{R_\theta} \approx \frac{0.441}{R_\theta}.$$

To a contrary with the Blasius equation, it was proved in [8] that solutions of (6) satisfying the physical boundary conditions form a two-parameter family. The parameters are R_θ, the Reynolds number based on momentum thickness and c_f, the skin-friction coefficient. The family of solutions matches experimental data for a wide range of Reynolds numbers, containing the transitional region and a part of the turbulent region of the boundary layer. The aim of this paper is to obtain a turbulent skin-friction law using the equation (6).

2. Velocity profiles

Note that if $\hat{h}(\xi)$ is a solution of (6), then $h(x) := \beta\hat{h}(\beta x)$ is a solution of

$$m''' + \frac{1}{2}hm'' = 0, \qquad m = h - h''. \tag{9}$$

The physical boundary conditions are $h(0) = h'(0) = 0$, $h''(0) > 0$, and $h'(x) \to \beta^2$ as $x \to \infty$.

The last boundary condition is equivalent to the condition that $\lim_{x\to\infty} h'(x)$ exists and satisfies

$$0 < \lim_{x\to\infty} h'(x) < \infty. \tag{10}$$

In this case h is a solution to our boundary value problem with β defined as

$$\beta := \left(\lim_{x\to\infty} h'(x)\right)^{1/2}.$$

In [8] we proved that the above boundary value problem posess a two parameter family. The parameters are

$$a := h''(0), \qquad b := h'''(0).$$

There is a function $C(a,b)$, which can be easily computed, such that for given $a > 0$ and b in a large open region, $h''''(0) = C(a,b)$ guarantees (10). More precisely, the following theorem holds:

Theorem 2.1. [8]. *There exists a continuous function $b_0 : (0,\infty) \to \mathbb{R}$ such that $b_0(a) < -a$, and for each $a > 0$ and $b \in (b_0(a), \infty)$ we have that $\lim_{x\to\infty} h'(x)$ exists and satisfies*

$$0 < \lim_{x\to\infty} h'(x) < \infty,$$

where $h(x)$ is a solution to (9) with $h(0) = h'(0) = 0$, $h''(0) = a$, $h'''(0) = b$, and $h''''(0) = C(a,b)$ for some function $C(a,b)$.

It is common to use the wall coordinates

$$y^+ = \frac{u_\tau y}{\nu}, \qquad u^+ = \frac{u}{u_\tau}$$

in the turbulent boundary layer, where

$$u_\tau = \sqrt{\frac{1}{\rho}\tau} = \sqrt{\nu\frac{\partial u}{\partial y}\Big|_{y=0}},$$

and τ is the shear stress at the wall.

Fix x_0 on the horizontal axis and denote

$$l_\mathrm{e} = \frac{\nu}{u_\mathrm{e}}, \qquad R_l = \frac{l}{l_\mathrm{e}},$$

where l is a parameter of the boundary layer at the point x_0. According to (7),

$$u(x_0, y) = \frac{u_\mathrm{e}}{\beta^2} h'\left(\frac{y}{\beta\sqrt{l_\mathrm{e} l}}\right) \tag{11}$$

represents the horizontal component of the averaged velocity at $x = x_0$ for some solution h of (9).

Note that (11) implies

$$a = h''(0) = \frac{1}{2}c_f\beta^3\sqrt{R_l},$$

i.e. a is a rescaled skin-friction coefficient $c_f = 2\left(u_\tau/u_\mathrm{e}\right)^2$. Writing (11) in wall coordinates, we obtain a three-parameter family of velocity profiles $u^+_{a,b,l}(\cdot)$:

$$u^+_{a,b,l}(y^+) = \frac{R_l^{1/4}}{\sqrt{a\beta}} h'\left(\frac{y^+\sqrt{\beta}}{R_l^{1/4}\sqrt{a}}\right). \tag{12}$$

In the following sections we will see that velocity profiles (12) satisfying the log low (they will be called the turbulent velocity profiles) form a one-parameter family $\{u^+\}_{R_\theta}$.

3. Von Karman log law

At very small distances from the wall, the viscous friction dominates shear stresses, and the velocity profile can be approximated by the following formula:

$$u^+ = y^+.$$

Further away from the boundary the Reynolds stresses can not be neglected, and the von Karman log law, or the logarithmic law of the wall

$$u^+ = \frac{1}{\kappa}\ln y^+ + B$$

is commonly used to approximate the mean velocity profile in the inner region of the flat plate turbulent boundary layer. This law is due L. Prandtl and T. von Karman (see [17]), and it has several semi-empirical derivations (see, e.g., [18]). However, recent experimental data has shown that this law is local rather than global. In order to reflect this locality, we use the following weaker formulation of this law, which is sufficient for our purposes:

Log Law.

(i) *A turbulent velocity profile $u_\mathrm{t}^+(y^+)$ has 3 inflection points in logarithmic coordinates.*

(ii) *The middle inflection point of $u_\mathrm{t}^+(y^+)$ lies on the line*

$$u^+ = \frac{1}{\kappa}\ln y^+ + B, \tag{13}$$

where κ and B are universal constants.

(iii) *The line (13) is tangent to $u_\mathrm{t}^+(y^+)$ at the middle inflection point.*

According to Landau and Lifshitz [18], the constants κ and B can not be determined theoretically and have to be obtained experimentally. There are numerous empirical estimates for of κ and B matching the log law with various experimental data, and the classical values used by Coles [11] are $\kappa = 0.41$, $B = 5$. We will assume the values $\kappa = 0.4$, $B = 5$, which fit the experimental data used in this study better.

We have shown numerically that the first part of the log law follows from the equation (6). A rigorous mathematical proof of this fact is not yet available.

In the next two sections we will study velocity profiles (12) subjected to the conditions of the log law. Velocity profiles satisfying conditions (i) and (ii) will be called transition velocity profiles. Velocity profiles satisfying conditions (i), (ii), and (iii) will be called turbulent velocity profiles.

4. Two-parameter family of transition velocity profiles

Transition velocity profiles satisfy

$$u^+_{a,b,l}(y_0^+) = \frac{1}{\kappa}\ln y_0^+ + B \tag{14}$$

for y_0^+, the middle inflection point in logarithmic coordinates. Note that (14) is equivalent to the condition that R_l solves the following equation:

$$\frac{R_l^{1/4}}{\sqrt{a\beta}} h'(\xi_0) = \frac{1}{\kappa}\ln(\sqrt{a/\beta}R_l^{1/4}\xi_0) + B. \tag{15}$$

Here ξ_0 is the middle inflection point of $h'(\xi)$ in logarithmic coordinates. This equation has two solutions for physically relevant values of a and b, and we discovered numerically that the largest one yields correct values of the skin friction coefficient. Therefore, the transition velocity profiles form a two-parameter family $\{u^+_{a,b}\}$ provided R_l is the largest solution to (15). The parameters a and b correspond to c_f, the skin-friction coefficient, and R_θ, the Reynolds number based on momentum thickness, which can be written in the following way:

$$c_f = \frac{2}{u^+(\infty)^2}, \tag{16}$$

$$R_\theta = \int_0^\infty u^+\left(1 - \frac{u^+}{u^+(\infty)}\right) dy^+, \tag{17}$$

where $u^+(\infty) := \lim_{y^+\to\infty} u^+$.

We determined numerically that for every pair of the physical parameters (R_θ, c_f) corresponding to transitional and turbulent boundary layers with $R_\theta < 3000$, there exists a pair of parameters (a, b) for which conditions (16, 17) hold. Varying R_θ and c_f, we obtained a family of the velocity profiles $\{u^+_{R_\theta,c_f}\}$. This family was compared with experimental data of the Rolls-Royce applied science laboratory, ERCOFTAC t3a and t3b test cases [22] (see Fig. 1–3).

5. Theoretical skin friction law and one-parameter family of turbulent velocity profiles

In this section we consider turbulent velocity profiles, i.e., profiles (12) subjected to all three conditions of the log law. In other words, we restrict the two-parameter family of velocity profiles $\{u^+_{R_\theta,c_f}\}$ obtained in the previous section to the condition (iii) of the log law. This condition connects the skin-friction coefficient with the Reynolds number based on momentum thickness and reduces the two-parameter family to just one-parameter family of velocity profiles $\{u^+_{R_\theta}\}$.

Let $F(z) = u^+_{R_\theta,c_f}(e^z)$. Since $F'(z) = e^z {u^+_{R_\theta,c_f}}'(e^z)$, condition (iii) of the log law can be written in the following way:

$$y^+_0 {u^+_{R_\theta,c_f}}'(y^+_0) = \frac{1}{\kappa}, \tag{18}$$

where y^+_0 is the middle inflection point of $u^+_{R_\theta,c_f}(y^+)$ in the logarithmic coordinates (see Section 3).

We determined numerically that for $350 \approx R^{\text{crit}}_\theta < R_\theta < 3000$, equation (18) has a unique solution $c_f = f(R_\theta)$. Therefore, for turbulent velocity profiles the skin-friction coefficient is a function of the Reynolds number based on momentum thickness The graph of this skin friction law $c_f = f(R_\theta)$ is shown on Fig. 4. This figure also shows skin friction coefficients from experiments of Roach and Brierlay (see [22]) with two different intensities of the free-stream turbulence. The Reynolds numbers based on momentum thickness were computed numerically according to formula (17).

At the critical point when $R_\theta = R^{\text{crit}}_\theta \approx 350$ and $c_f = c^{\text{crit}}_f \approx 0.0063$, the second and the third inflection points collapse, and the velocity profile starts having only one inflection point for $R_\theta \le R^{\text{crit}}_\theta$. Therefore our model yields that R^{crit}_θ is the minimal value of the Reynolds number based on momentum thickness for which a velocity profile can be turbulent. In addition, c^{crit}_f is the largest value of the skin friction coefficient for a turbulent velocity profile. However, we expect existence of transitional profiles with a skin-friction coefficient slightly larger than c^{crit}_f.

6. Conclusion

The Navier-Stokes-α and Leray-α models are of higher order than the Navier-Stokes equations and hence they require additional boundary conditions. To determine the unknown boundary conditions, we proposed to use physical parameters of the flow near the boundary. Another approach, due to Marsden and Shkoller, is to use anisotropic equations that degenerate on the boundary (see [19, 12, 13]). This requires, however, introducing a weighting function in the viscous sublayer. A comparison of solutions to these equations with experimental data will be of great interest.

From the Leray-α model we derive a generalized Falkner-Skan equation:

$$m''' + \tfrac{1}{2}hm'' + \lambda(1 - h'm') = 0, \qquad m = h - \beta h''.$$

It is remarkable that the Navier-Stokes-α model can be reduced to to this equation only in the zero-pressure gradient case, i.e., when $\lambda = 0$. In this case both models give the same result, and the only difference is that the pressure term q in the Navier-Stokes-α model has to be modified in order to obtain the physical averaged pressure. It is clear how to do this in a boundary layer (see [8]), but not in general. To a contrary, the pressure term p in the Leray-α model corresponds to the physical averaged pressure, which makes the analysis a little easier. The case $\lambda \neq 0$, which can help us better understand differences between the two α-models, is currently under the investigation.

In the Blasius case $\lambda = 0$, it looks like there are three additional free parameters: $h'''(0)$, $h''''(0)$, and β. However, it was shown in [8] that almost all the solutions blow-up, and there exists only a two-parameter family of solutions satisfying the non-slip boundary conditions. This manifold of solutions consists of transitional and turbulent velocity profiles, where the parameters are the Reynolds number and the skin-friction coefficient. In order to isolate only turbulent velocity profiles, one has to specify an additional condition. Since all experimental turbulent velocity profiles have the same tangent line at the middle inflection point, we use this weak form of the von Karman Log Law as a such condition. As we found out, there is only a one-parameter family of turbulent velocity profiles, which we compare with experimental data.

Finally, we would like to emphasize that the only free parameter for turbulent velocity profiles is the Reynolds number, which is consistent with experimental data. All the other quantities (e.g., the skin-friction coefficient) are functions of the Reynolds number and can be explicitly computed. The graph of the skin-friction coefficient is presented on Fig. 4.

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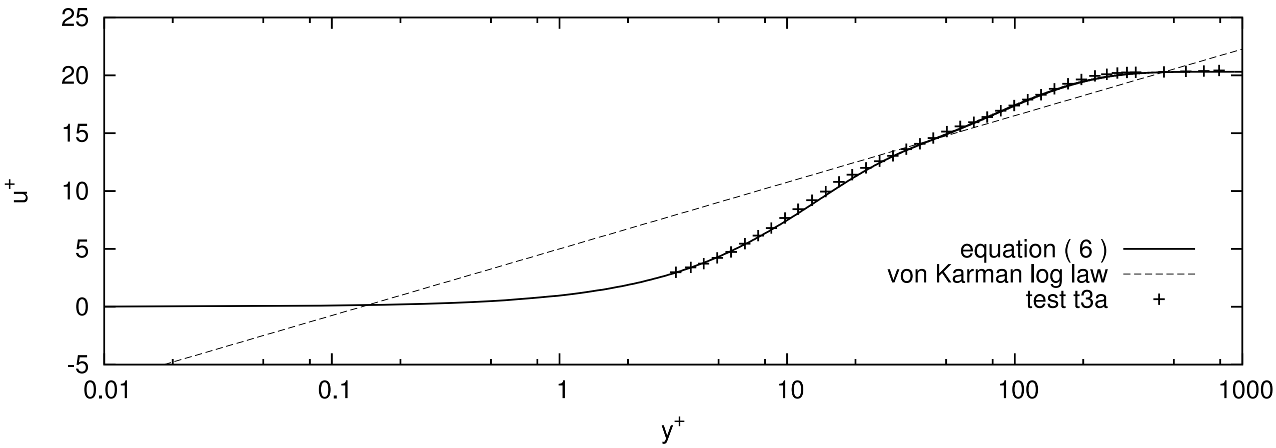

FIGURE 1. Comparison with experimental data for $c_f = 0.0048,\ R_\theta = 603$

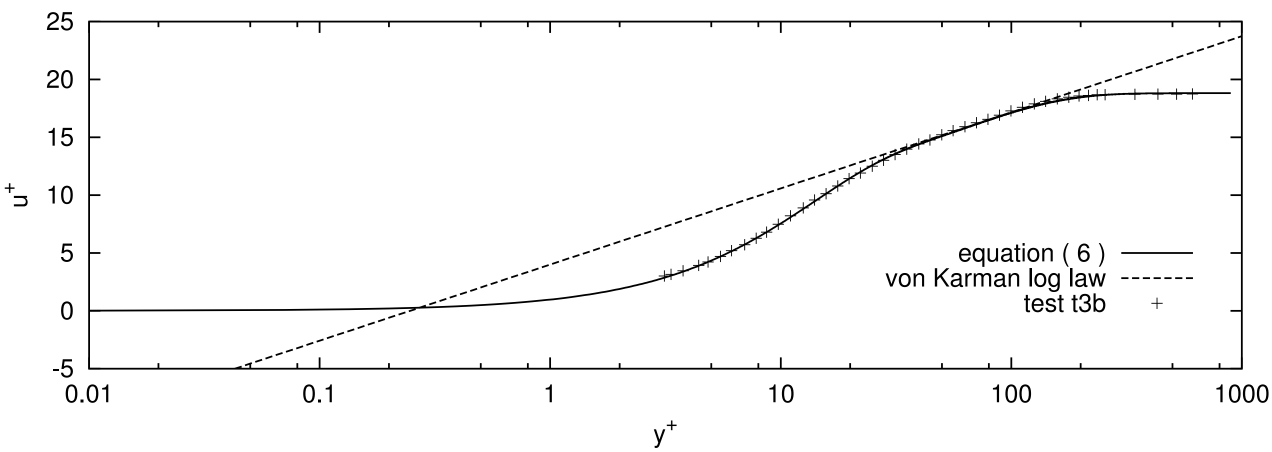

FIGURE 2. Comparison with experimental data for $c_f = 0.00569,\ R_\theta = 396$

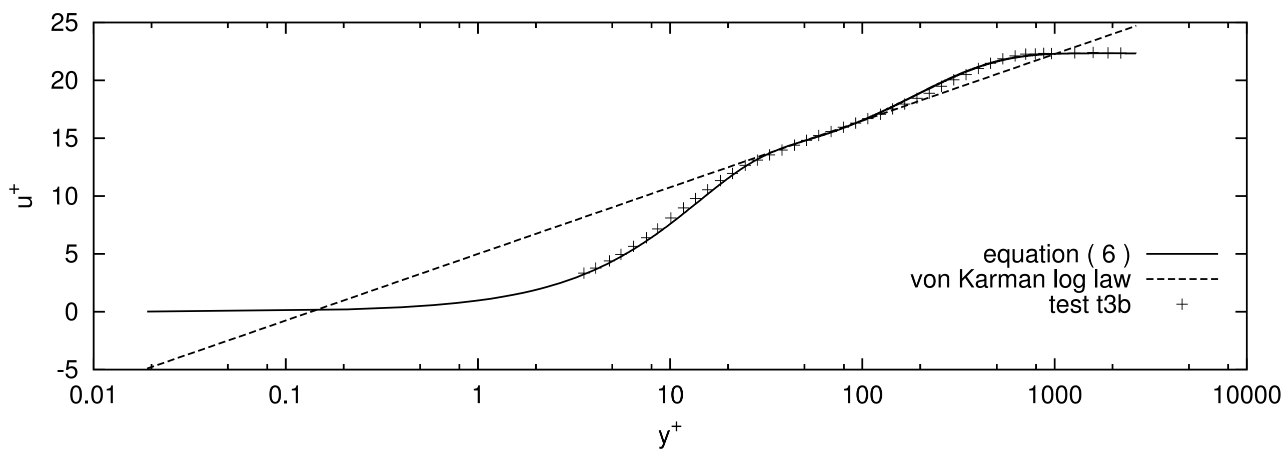

FIGURE 3. Comparison with experimental data for $c_f = 0.00401,\ R_\theta = 1436$

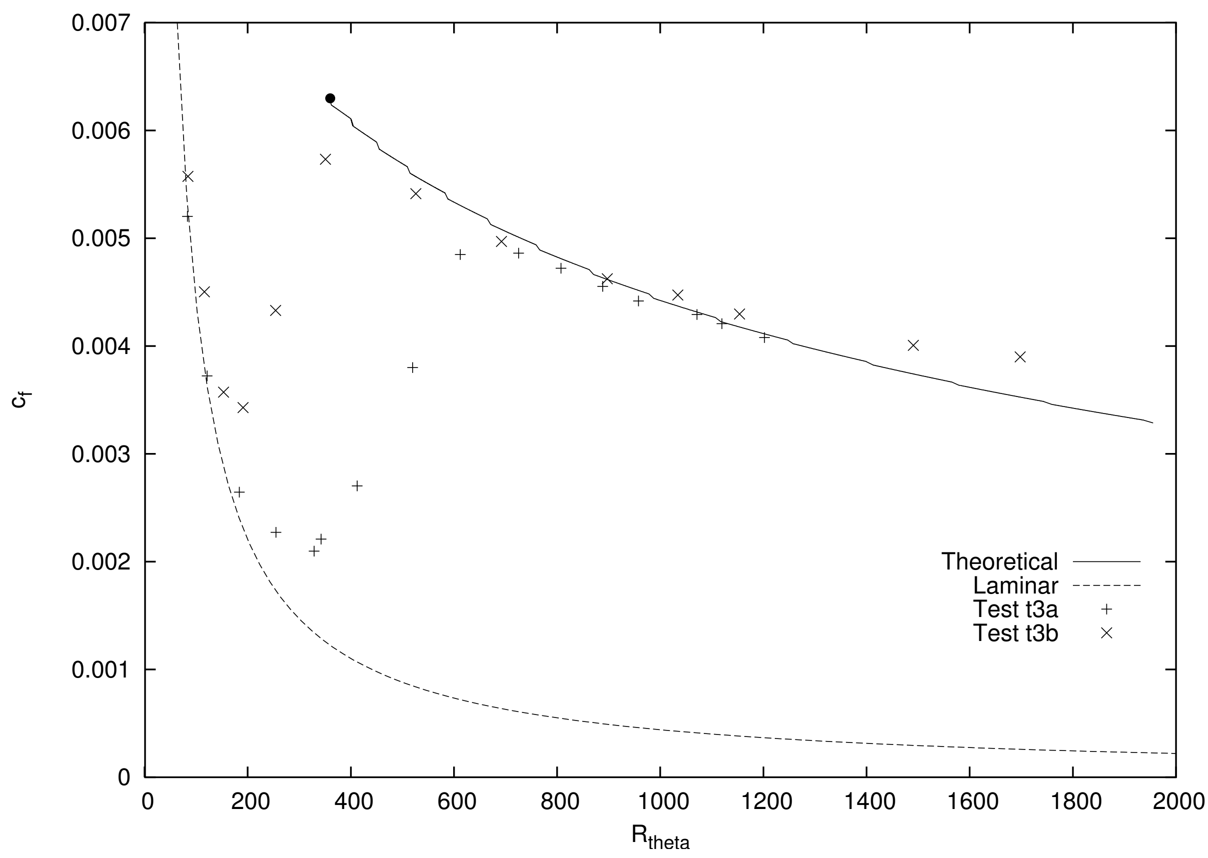

FIGURE 4. Theoretical skin-friction law