Final, due 5/9, Math 446, professor Agol, winter 2002

You may use any references you wish, but do not collaborate or discuss the problems with anyone else in the class (although you should ask Agol questions about the problems if they don't make sense to you).

1. Compute the abelianization $\pi_1(S)^{ab}$ for non-orientable surfaces. Show that this distinguishes the homeomorphism types of closed connected surfaces with distinct normal forms determined in 1.3.7.

2. Use the fundamental group, number of boundary components, and orientability type to classify compact connected surfaces with boundary (Hint: for the orientability part, cap off boundary components of the surface with disks to get a closed surface of the same orientability type, and use problem 1). The point of this question is that we never proved invariance of the Euler characteristic under homeomorphism of surfaces.

3. Recall that for a subsimplex $\Delta^k \subset \Delta^n$, $\mathcal{N}_\epsilon(\Delta^k, \Delta^n) = \{\sum_{i=0}^n \alpha_i e_i \mid \sum_{i=0}^n \alpha_i = 1,\ \alpha_i \geq 0, \sum_{i=k+1}^n \alpha_i < \epsilon\}$, where we assume $0 < \epsilon < 1$, and e_i is the elementary vector in $\mathbb{R}^{n+1}$ with a 1 in the ith spot, and 0 at every other coordinate, Δ^k is spanned by $e_0, ..., e_k$. (If $k = n$, then $\mathcal{N}_\epsilon(\Delta^n, \Delta^n) = \Delta^n$, since the second sum is trivial.)

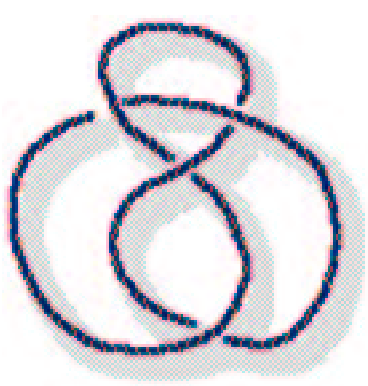
If $\mathcal{C} \subset \mathcal{D}$ is a subcomplex $\mathcal{C}$ of a simplicial complex $\mathcal{D}$, then

$$\mathcal{N}_\epsilon(\mathcal{C}, \mathcal{D}) = \cup_{\Delta_1 \subset \mathcal{C}, \Delta_2 \subset \mathcal{D}, \Delta_1 \subset \Delta_2} \mathcal{N}_\epsilon(\Delta_1, \Delta_2).$$

Show that $\mathcal{N}_\epsilon(\mathcal{C}, \mathcal{D})$ deformation retracts to $\mathcal{C}$.

4. Prove Hall's theorem: Let F be a free group with finitely many generators, and $H < F$ a finitely generated subgroup. Show that there is a finite index subgroup $\tilde{F} \leq F$, such that $H \leq \tilde{F}$, and $\tilde{F} \cong H * H'$. (Possible approach: take a bouquet of circles B such that $\pi_1(B) = F$, and take a cover B' of B, such that $\pi_1(B') = H$. Then find a finite subgraph $G \subset B'$ such that $\pi_1(G) = \pi_1(B')$. Then figure out how to find a finite sheeted $B' \to \tilde{B} \to B$ such that $G \to \tilde{B}$ is an embedding.)

5. Compute a Wirtinger presentation for the figure eight knot complement, and reduce the



presentation using Tietze transformations to one with two generators and one relation.

6. Prove that there is no retraction $r : D \times S^1 \to \partial(D \times S^1)$.

7. Prove that if a map $f : S^2 \to S^2$ has no fixed points, then it is homotopic to the antipodal map $a : S^2 \to S^2$, $a(x) = -x$.