

Final solutions, Math 446, professor Agol, winter 2003

You may use any references you wish, but do not collaborate or discuss the problems with anyone else in the class (although you should ask Agol questions about the problems if they don't make sense to you).

1. Compute the abelianization $\pi_1(S)^{ab}$ for non-orientable surfaces. Show that this distinguishes the homeomorphism types of closed connected surfaces with distinct normal forms determined in 1.3.7.

By 1.3.7, any non-orientable closed surface P_n with n crosscaps has a normal form denoted symbolically by $a_1^2 a_2^2 \cdots a_n^2$. This represents a 2-complex with one 2-cell, n edges and one vertex. By 3.4.4, the fundamental group has a presentation $\pi_1(P_n) \cong \langle a_1, \dots, a_n \mid a_1^2 \cdots a_n^2 \rangle$. The abelianization is given by $\pi_1(P_n)^{ab} \cong \langle a_1, \dots, a_n \mid a_1^2 \cdots a_n^2, a_i a_j = a_j a_i, 1 \leq i, j \leq n \rangle = \langle a_1, \dots, a_n \mid a_i a_j = a_j a_i, (a_1 \cdots a_n)^2 \rangle$, since the fact that $a_i a_j = a_j a_i$ means that we may write the generators in a relator in any order we like (this is proved easily by induction). Let $u = a_1 \cdots a_n$, then $a_n = a_{n-1}^{-1} \cdots a_1^{-1} u$, so we may use Tietze transformations to replace a_n with u . Thus $\pi_1^{ab} \cong \langle a_1, \dots, a_{n-1}, u \mid u^2 = 1, a_i a_j = a_j a_i, u a_j = a_j u \rangle \cong \mathbb{Z}^{n-1} \times (\mathbb{Z}/2\mathbb{Z})$. Now, if $P_n \cong P_m$, then we see that $\mathbb{Z}^{n-1} \times (\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}^{m-1} \times (\mathbb{Z}/2\mathbb{Z})$, which is only possible if $n = m$ (this follows from the classification of finitely generated abelian groups, but may be seen by tensoring with \mathbb{Q} or with \mathbb{F}_2 , and comparing ranks of the resulting vector spaces). Also, since $\pi_1(P_n)^{ab}$ has torsion (since $u^2 = 1$), but $\pi_1(S_g) = \mathbb{Z}^{2g}$ does not (S_g is the orientable surface of genus g), we see that none of the non-orientable surfaces P_n are homeomorphic to orientable surfaces S_g .

2. Use the fundamental group, number of boundary components, and orientability type to classify compact connected surfaces with boundary (Hint: for the orientability part, cap off boundary components of the surface with disks to get a closed surface of the same orientability type, and use problem 1). The point of this question is that we never proved invariance of the Euler characteristic under homeomorphism of surfaces.

Suppose we have two surfaces S_1 and S_2 , such that $S_1 \cong S_2$. Then $\partial S_1 \cong \partial S_2$, so S_1 and S_2 must have the same number of boundary components. Let \hat{S}_i be the surface obtained by adding disks to all the boundary components of S_i . If we have a homeomorphism $h : S^1 \rightarrow S^1$, then we get a homeomorphism $H : D \rightarrow D$ given by $H(re^{it}) = rh(e^{it})$ so that $H|_{S^1} = h$. Thus, if we have a homeomorphism $h : S_1 \rightarrow S_2$, then we may extend this to a homeomorphism $H : \hat{S}_1 \rightarrow \hat{S}_2$. Now, the normal forms for surfaces with boundary were given in section 1.3.9. If \hat{S}_1 is orientable, and \hat{S}_2 is non-orientable, then $\hat{S}_1 \not\cong \hat{S}_2$ by the previous question, so $S_1 \not\cong S_2$. Thus, \hat{S}_1 and \hat{S}_2 must be simultaneously orientable or non-orientable. From the normal forms given in figure 99.(i), we see that if \hat{S}_i are orientable of genus g , and have n boundary components, then $\pi_1(S_i) \cong F_{2g+n-1}$, the free group on $2g + n - 1$ generators. If we have a disk with n disks removed, then the fundamental group is the free group $\langle w_1, \dots, w_n \rangle$, where w_i is a loop going around the i th boundary, and the boundary of the disk is represented by the word $w_1 w_2 \cdots w_n$. If we have a surface with one boundary component and genus g , then it collapses to the graph with fundamental group $\langle a_1, b_1, \dots, a_g, b_g \rangle$ such that the boundary is represented by $[a_1, b_1] \cdots [a_g, b_g]$. Then gluing the n -holed disk to the genus g surface with one boundary, we get a genus g surface with n boundary components, which by Seifert-Van Kampen has presentation $\langle w_1, \dots, w_n, a_1, b_1, \dots, a_g, b_g \mid w_1 \cdots w_n = [a_1, b_1] \cdots [a_g, b_g] \rangle \cong \langle w_1, \dots, w_{n-1}, a_1, b_1, \dots, a_g, b_g \rangle$ since we may use a Tietze transformation to get rid of the generator w_n and the relator. Thus, two orientable surfaces with the same fundamental group and number of boundary components have the same genus, since we can obtain this from the rank of the free group, and are therefore homeomorphic. If a surface is non-orientable with p cross-caps, and n boundary components, then the fundamental group is $\langle w_1, \dots, w_n, a_1, \dots, a_p \mid a_1^2 a_2^2 \cdots a_p^2 = w_1 \cdots w_n \rangle \cong \langle w_1, \dots, w_{n-1}, a_1, \dots, a_p \rangle$, so the fundamental group has rank $p + n - 1$, and we see

that we may derive the cross-cap number from the rank of the fundamental group and the number of boundary components. Thus, two non-orientable surfaces with the same number of boundary components are homeomorphic if and only if they have the same fundamental group and the same cross-cap number.

3. Recall that for a subsimplex $\Delta^k \subset \Delta^n$, $\mathcal{N}_\epsilon(\Delta^k, \Delta^n) = \{\sum_{i=0}^n \alpha_i e_i \mid \sum_{i=0}^n \alpha_i = 1, \alpha_i \geq 0, \sum_{i=k+1}^n \alpha_i < \epsilon\}$, where we assume $0 < \epsilon < \frac{1}{2}$, and e_i is the elementary vector in \mathbb{R}^{n+1} with a 1 in the i th spot, and 0 at every other coordinate, Δ^k is spanned by e_0, \dots, e_k . (If $k = n$, then $\mathcal{N}_\epsilon(\Delta^n, \Delta^n) = \Delta^n$, since the second sum is trivial.)

If $\mathcal{C} \subset \mathcal{D}$ is a subcomplex \mathcal{C} of a simplicial complex \mathcal{D} , then

$$\mathcal{N}_\epsilon(\mathcal{C}, \mathcal{D}) = \cup_{\Delta_1 \subset \mathcal{C}, \Delta_2 \subset \mathcal{D}, \Delta_1 \subset \Delta_2} \mathcal{N}_\epsilon(\Delta_1, \Delta_2).$$

Show that $\mathcal{N}_\epsilon(\mathcal{C}, \mathcal{D})$ deformation retracts to \mathcal{C} .

This problem is still not correct. A counterexample is given by $\mathcal{C} = \partial\Delta^2$, $\mathcal{D} = \Delta^2$, then $\mathcal{N}_{\frac{1}{2}}(\mathcal{C}, \mathcal{D}) = \mathcal{D} - \frac{1}{2}\{e_0 + e_1, e_1 + e_2, e_0 + e_2\}$, but there is no retraction $\mathcal{N}_{\frac{1}{2}}(\mathcal{C}, \mathcal{D}) \rightarrow \partial\Delta^2$.

A better way to choose a regular neighborhood is to take the 2nd barycentric subdivision of \mathcal{D} , and choose all open simplices of this which are adjacent to simplices of \mathcal{C} . Then this gives an open neighborhood of \mathcal{C} in \mathcal{D} which deformation retracts to \mathcal{C} .

4. Prove Hall's theorem: Let F be a free group with finitely many generators, and $H < F$ a finitely generated subgroup. Show that there is a finite index subgroup $\tilde{F} \leq F$, such that $H \leq \tilde{F}$, and $\tilde{F} \cong H * H'$. (Possible approach: take a bouquet of circles B such that $\pi_1(B) = F$, and take a cover B' of B , such that $\pi_1(B') = H$. Then find a finite subgraph $G \subset B'$ such that $\pi_1(G) = \pi_1(B')$. Then figure out how to find a finite sheeted $B' \rightarrow \tilde{B} \rightarrow B$ such that $G \rightarrow \tilde{B}$ is an embedding.)

Let B be a bouquet of circles with vertex P such that $\pi_1(B, P) = F$. Let $H < F$ be a finitely generated subgroup. By 2.2.2, there is a covering $\phi : (B', \tilde{P}) \rightarrow (B, P)$ such that $\phi_* \pi_1(B', \tilde{P}) = H$. We may take a spanning tree \mathcal{T} of B' , and by 2.1.7, for each edge $e_i \in B' - \mathcal{T}$, there is an associated generator $a_i = w_j e_i w_k^{-1}$, so that the set $\{a_i\}$ freely generate $\pi_1(B', \tilde{P})$. Since H is finitely generated, the set $\{a_i\}$ must be a finite free generating set (we showed in class that any free generating set has the same cardinality, by considering the rank of the abelianization).

Taking the union of all the e_i and w_j 's with endpoint on an edge e_i , we get a connected subgraph $G \subset B'$ such that $\pi_1(G, P) \cong \pi_1(B', P)$ (in fact, there is a minimal finite subgraph carrying H , by taking the intersection over all such subgraphs). Now, suppose e is an edge of $B' - G$ such that e has an endpoint $v \in G$, and e is labelled by a generator g_i^ϵ of F , $\epsilon = \pm 1$, oriented away from v . Then there should be another vertex $w \in G$ and an edge $f \in B' - G$ such that f is labelled by $g_i^{-\epsilon}$ oriented away from w . To see this, consider the subgraph G_i of G consisting of all vertices of G and all edges labelled by g_i . Every vertex of G_i has either zero, one or two edges adjacent, so it consists of paths (where a single vertex is considered a path of length 0) or loops. The connected component of G_i containing v must be a path, so the other endpoint of the path must be a vertex w of degree 1 in G_i , which therefore has an adjacent edge labelled $g_i^{-\epsilon}$. Thus, for each path component of each G_i , we may add an edge labelled g_i to G connecting its two endpoints, to give us a finite graph \tilde{B} containing B' as a subgraph, with vertex set the same as B' , such that each vertex has an adjacent edge labelled by every generator of F , which is therefore a finite-sheeted covering graph of B . We may consider $\mathcal{T}' = \mathcal{T} \cap B'$, which is a spanning tree for B' , but also for \tilde{B} . Then H is the subgroup corresponding to edges in $B' - \mathcal{T}'$, and H' the subgroup corresponding to edges of $\tilde{B} - B'$. Since together these edges give free generators of $\pi_1(\tilde{B}, P)$, we see that $\pi_1(\tilde{B}, P) = F' = H * H'$, as desired.

5. Compute a Wirtinger presentation for the figure eight knot complement, and reduce the presentation using Tietze transformations to one with two generators and one relation.



Using Wirtinger's presentation, we get a presentation

$$\langle a_1, a_2, a_3, a_4 \mid a_1 a_4 = a_4 a_2, a_2 a_4 = a_3 a_2, a_3 a_4 = a_1 a_3, a_3 a_1 = a_1 a_2 \rangle.$$

The last relator gives us $a_3 = a_1 a_2 a_1^{-1}$, so we may substitute this in to the other relators to obtain relators which we may add in by T_1 move $a_2 a_4 = a_1 a_2 a_1^{-1} a_2$ and $a_1 a_2 a_1^{-1} a_4 = a_1 a_1 a_2 a_1^{-1}$, then use these to eliminate the 2nd and 3rd relators by a T_1^{-1} move, since these are consequences of the new relators and the 4th relator. Then we may use a T_2^{-1} move to get rid of the relator $a_3 a_1 = a_1 a_2$ and the generator a_3 , to get a presentation $\langle a_1, a_2, a_4 \mid a_1 a_4 = a_4 a_2, a_2 a_4 = a_1 a_2 a_1^{-1} a_2, a_2 a_1^{-1} a_4 = a_1 a_2 a_1^{-1} \rangle$. Now we use the relator $a_4 = a_2^{-1} a_1 a_2 a_1^{-1} a_2$ to get rid of a_4 , in a similar fashion, substituting this in for other occurrences of a_4 and getting rid of this relator: $\langle a_1, a_2 \mid a_1 a_2^{-1} a_1 a_2 a_1^{-1} = a_2^{-1} a_1 a_2 a_1^{-1} a_2, a_2 a_1^{-1} a_2^{-1} a_1 a_2 a_1^{-1} a_2 = a_1 a_2 a_1^{-1} \rangle$, and by multiplying the second relator by $a_1 a_2^{-1}$, we see that these two relators are equivalent, so we obtain the presentation $\langle a_1, a_2 \mid a_1 a_2^{-1} a_1 a_2 a_1^{-1} = a_2^{-1} a_1 a_2 a_1^{-1} a_2 \rangle$.

6. Prove that there is no retraction $r : D \times S^1 \rightarrow \partial(D \times S^1)$.

$\partial(D \times S^1) \cong S^1 \times S^1$. $\pi_1(D \times S^1) \cong \pi_1(D) \times \pi_1(S^1) \cong 1 \times \mathbb{Z} \cong \mathbb{Z}$. $\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$. If we had a retraction $r : D \times S^1 \rightarrow \partial(D \times S^1)$, this would induce an onto map $r_* : \pi_1(D \times S^1) \rightarrow \pi_1(S^1 \times S^1)$, so we would get an onto map $r_* : \mathbb{Z} \rightarrow \mathbb{Z} \times \mathbb{Z}$. This is impossible, since $r_*(\mathbb{Z}) = \mathbb{Z}(a, b)$, but $(-b, a) \notin \mathbb{Z}(a, b)$, since then $-b = na, a = nb$, so $-b = n^2 b$, and $b = 0$, so $a = nb = 0$, so $a = b = 0$. But if $a = b = 0$, then $(1, 0) \notin \mathbb{Z}(a, b)$, a contradiction.

7. Prove that if a map $f : S^2 \rightarrow S^2$ has no fixed points, then it is homotopic to the antipodal map $a : S^2 \rightarrow S^2$, $a(x) = -x$.

Let $h(x, t) = ((1-t)f(x) - tx) / \|(1-t)f(x) - tx\|$. Since $f(x) \neq x$, the line between $f(x)$ and $-x$ in \mathbb{R}^3 will not pass through the origin, so $(1-t)f(x) - tx \neq 0$, and we see that $h(x, t)$ is well-defined and continuous. Also, $h : S^2 \times [0, 1] \rightarrow S^2$, since we have normalized the length to be 1. $h(x, 0) = f(x)$, and $h(x, 1) = -x = a(x)$, so h is a homotopy between f and a .