Homework 10 solutions, Math 446, professor Agol, winter 2002

1.3.8.2 Let $\mathcal{F}$ be an actual euclidean polyhedral surface. Define the *curvature* $\kappa(P)$ of $\mathcal{F}$ at a vertex P to be (2π minus the sum of the face angles incident with P). Then show that

$$\sum_{\text{vertices } P} \kappa(P) = 2\pi \times (\text{Euler characteristic of } \mathcal{F}).$$

Each face of $\mathcal{F}$ is a euclidean polygon. Thus, if F is a face with n edges, then the sum of the exterior angles is 2π, and the sum of the interior angles is $n\pi - 2\pi$. So each face F contributes $n\pi - 2\pi$ to the total angle sum. Let $\alpha(P, F)$ denote the dihedral angle of the face F at the vertex P of F, and let f_n denote the number of faces of degree n. We have $\chi(\mathcal{F}) = v - e + f$, where v is the number of vertices, e is the number of edges, and f is the number of faces. Each edge is contained in exactly two faces, so we get $\sum_n n f_n = 2e$. Then we have

$$\sum_P \kappa(P) = \sum_P (2\pi - \sum_{F \ni P} \alpha(P, F)) = \sum_P 2\pi - \sum_F \sum_{P \in F} \alpha(P, F) = 2\pi v - \sum_F (\text{degree}(F)\pi - 2\pi)$$

$$= 2\pi v - \pi \sum_n n f_n + 2\pi f = 2\pi(v - e + f) = 2\pi\chi(\mathcal{F}).$$

1.3.9.1 Show that the normal forms of bounded surfaces are distinguished from each other by Euler characteristic, orientability character, and number of boundary components.

We are assuming here that homeomorphic surfaces have the same Euler characteristic. Certainly, if two surfaces are homeomorphic, then their boundaries are homeomorphic, so in particular, the number of boundary curves must be the same. Given two orientable surfaces with the same number of boundary components, the Euler characteristics are $2 - 2g - c$, where g is the genus, and c is the number of boundary components. So they are homeomorphic if and only if the genus is the same. For a non-orientable surface, the Euler characteristic is $2 - p - c$, where p is the number of cross-caps, and c is the number of boundary components, so two non-orientable surfaces with the same number of boundary components must have the same number of cross-caps.

1.3.9.3 One way to see that the standard surface may be turned inside out is to add handles to a disk. Then one can push each handle "through" the surface, until the hole becomes a handle on the other side. Then rotate the handle by $\pi/2$, and flip the whole thing over, to get back to the original surface turned inside out.

1.4.1.1 Associate each sheet in the obvious way with an ordered pair $\langle m, n\rangle$ of integers $m, n \in \mathbb{Z}$. Then describe the permutations of the sheets induced by crossing the lines a, b on the torus as permutations of $\mathbb{Z} \times \mathbb{Z}$.

Associate to each region (in figure 104) the coordinate of its center, normalized so that these are integers. Then crossing a corresponds to the permutation $(m, n) \mapsto (m + 1, n)$, and crossing b corresponds to the permutation $(m, n) \mapsto (m, n + 1)$. Together, these generate the group $\mathbb{Z} \times \mathbb{Z}$ acting on itself, with generators $(1, 0)$ and $(0, 1)$.

1.4.2.1 By computing the Euler characteristic of the n-sheeted cover of $\mathcal{F}_2$, show that it can be an orientable surface of arbitrary genus > 1, if n is suitably chosen.

Let S_n denote the n-fold cover of $\mathcal{F}_2$, as described in the problem. Since the Euler characteristic is additive under gluing along circles (which have $\chi(S^1) = 0$), $\chi(S_n) = n\chi(\mathcal{F}_2) = -2n = 2 - 2g$. Thus, letting $g = 1 + n$, we see that we may get arbitrary genus > 1.