Homework 11 solutions, Math 446, professor Agol, winter 2002

2.1.3.2 This problem is incorrect. One way to rectify it would be to use an octagon, with corners of dihedral angles $\pi/4$. The hyperbolic plane may be tiled by these, giving a cell decomposition of the plane, and taking alternating edges in the 1-skeleton, one obtains a graph which is a tree. The symmetries of this graph in the plane (preserving a certain labelling) give a free group with two generators acting freely on the hyperbolic plane. One may take free subgroups generated freely by n elements of the free group with 2 generators, giving actions by isometries of the hyperbolic plane.

2.1.5.2 Show that the 2-dimensional lattice graph (figure 123) has a spanning tree homeomorphic to the real line (that is, every vertex meets two edges).

One could take a spiral starting at the origin.

2.1.6.1 Show that we are entitled to omit mention of P in the notation for the fundamental group of a connected graph by proving that the choice of any other vertex P' as the origin for closed paths leads to an isomorphic group.

Let p be a closed path based at the vertex P, representing an element of $\pi_1(G,P)$. Let r be a path going from P to P', which exists since G is assumed to be connected. Define a map $f:\pi_1(G,P)\to\pi_1(G,P')$ by $f([p])=[r^{-1}pr]$. First, we need to show that this is well-defined. $r^{-1}pr$ is a closed path at P. If $[p]=[q]$, then p and q are related by adding or deleting spurs, and thus $r^{-1}pr$ and $r^{-1}qr$ will be equivalent. Second, we claim f is a homomorphism. $f([p])f([q])=[r^{-1}pr][r^{-1}qr]=[r^{-1}prr^{-1}qr]=[r^{-1}pqr]=f([pq])$, since rr^{-1} is reducible to the trivial word by eliminating spurs. Similarly, we have a homomorphism $g:\pi_1(G,P')\to\pi_1(G,P)$, defined by $g([p'])=[rp'r^{-1}]$. Then $g\circ f=id$, since $g\circ f([p])=g([r^{-1}pr])=[rr^{-1}prr^{-1}]=[p]$, and similarly $f\circ g=id$. Thus, g is the inverse to f, and f and g are isomorphisms.

2.1.7.1 Show that the number of edges not in a spanning tree $\mathcal{T}$ of $\mathcal{G}$ is independent of the choice of $\mathcal{T}$.

Consider two spanning trees $\mathcal{T}_1$ and $\mathcal{T}_2$. Let $e_1\in\mathcal{T}_1-\mathcal{T}_2$. If no such e exists, then $\mathcal{T}_1\subset\mathcal{T}_2$. But then $\mathcal{T}_1=\mathcal{T}_2$, otherwise there would be an edge $e_2\in\mathcal{T}_2-\mathcal{T}_1$, and the endpoints of e_2 would be connected by a reduced path in $\mathcal{T}_1$, forming a closed reduced path in $\mathcal{T}_2$, a contradiction. Thus, if $\mathcal{T}_1\neq\mathcal{T}_2$, then there is $e_1\in\mathcal{T}_1-\mathcal{T}_2$. Then end points of e_1 are connected by a reduced path p in $\mathcal{T}_2$. Choose an edge f of $p-\mathcal{T}_1$, which must exist, otherwise $p\cup e_1$ would be a closed reduced path in T_1. Then $\mathcal{T}'=\mathcal{T}_2-f\cup e_1$ is a new tree. It is easy to check that $\mathcal{T}'$ is connected. Also, $\mathcal{T}'$ has no closed reduced loops. If it did, then the closed reduced loop would have to contain e_1, otherwise it would be a closed reduced loop in $\mathcal{T}_2$, a contradiction. But if the closed loop contained e_1, then the rest of the loop would form a reduced path in $\mathcal{T}_2$ connecting the endpoints of e_1, and thus must be p by the uniqueness of reduced paths in trees. But $\mathcal{T}'$ does not contain f, so could not contain p, a contradiction. Thus, $\mathcal{T}'$ is a spanning tree. We see that the number of edges in $\mathcal{G}-\mathcal{T}'$ is the same as that in $\mathcal{T}_2$, and $\mathcal{T}_1-\mathcal{T}_2$ has one fewer edge. Inducting on the number of edges in $\mathcal{T}_1-\mathcal{T}_2$, we see that the number of edges outside of these two trees are equal. If $\mathcal{G}$ is infinite, then there still might be finitely many edges in $\mathcal{G}-\mathcal{T}_1$, in which case the same argument works to show that there are the same number of edges outside of $\mathcal{G}-\mathcal{T}_2$. Thus, the number of edges in $\mathcal{G}-\mathcal{T}_1$ and $\mathcal{G}-\mathcal{T}_2$ are either both finite (and equal) or both infinite.

Find the general formula for an isometry of the hyperbolic plane (in the conformal unit disk model).

Complex conjugation $z\mapsto\overline{z}$ is an isometry, which reverses orientation. The orientation preserving isometries are given by

$$e^{i\phi}\frac{z-\alpha}{\overline{\alpha}z-1},$$

where $\phi\in\mathbb{R}$, and $|\alpha|<1$. The orientation reversing isometries are obtained by composing these with complex conjugation. To show this, one observes that these maps are isometries of the unit disk, and are closed under composition. Also, one may clearly take any point α to 0, and one may rotate about 0 using $e^{i\phi}$. Then one checks that any orientation preserving isometry is determined by where it takes a unit vector at the origin, and notices that the given isometries are transitive on unit vectors.