Homework 12 solutions, Math 446, professor Agol, winter 2002

2.2.6.1 Use the Schreier method to find free generators for the commutator subgroup K_2 of F_2.

In the covering graph corresponding to the subgroup K_2, shown in figure 128, take a spanning tree consisting of the real axis and all the vertical lines with integral coefficients (that is, all paths of the form $e_2^n e_1^m, n, m \in \mathbb{Z}$. Then the generators are of the form $e_2^n e_1^m e_2 e_1^{-m} e_2^{n-1}$.

2.2.7.1 A covering $\tilde{\mathcal{G}}$ of $\mathcal{G}$ is called *regular* if the paths in $\tilde{\mathcal{G}}$ which cover a given closed path p in $\mathcal{G}$ are either all closed or all nonclosed. Show that this property is equivalent to the normality of the subgroup realized by the covering.

If the subgroup is normal $G \lhd F$, then $\tilde{\mathcal{G}}$ is the Cayley graph of F/G. So any two lifts of a path in $\mathcal{G}$ will represent multiplication by the corresponding group element of F/G, so the path will be closed iff the group element in F/G is trivial.

Conversely, suppose we have a cover $\tilde{\mathcal{G}}$ such that for any closed path p in $\mathcal{G}$, every path in $\tilde{\mathcal{G}}$ covering p is either closed or nonclosed. Since vertices of $\tilde{\mathcal{G}}$ correspond to right cosets F/G (see 2.2.3), then a path p representing an element of F acts on right cosets by multiplication. For a path p representing an element of G, we see $Gp = G$, so p lifts to a closed path starting at the vertex G, so $Gfp = Gf$ for all $f \in F$, so $fpf^{-1} \in G$, for all f, and we see that G is normal.

2.2.7.3 Identify the normal subgroup G of F_2 realized by the covering in Figure 131 and the quotient F_2/G. Give a set of free generators for G.

A set of Schreier transversals is given by $e_1^n, n \in \mathbb{Z}$, and we see that generators are given by $e_1^n e_2 e_1^{-n}$. This is the normal subgroup of all words with the exponent sum of $e_1 = 0$. The quotient group $F_2/G \cong \langle e_1 \rangle$, since this group has presentation $\langle e_1, e_2 | e_2 \rangle \cong \langle e_1 \rangle$.

Construct the cover of the bouquet of two circles with fundamental group $F = \langle a, b \rangle$, corresponding to the subgroup $G = \langle g^2, g \in F \rangle$.

It is clear that $G \lhd F$, since $hg^2h^{-1} = (hgh^{-1})^2 \in G$. Then $F/G \cong (\mathbb{Z}/2\mathbb{Z})^2$, since $a^2 = b^2 = 1 \in F/G$, and $(ab)^2 = abab = aba^{-1}b^{-1} = 1 \in F/G$. Thus, the cover will have four vertices, corresponding to the cosets G, Ga, Gb, Gab, connected as the Cayley graph of $(\mathbb{Z}/2\mathbb{Z})^2 \cong \langle a, b | a^2, b^2, (ab)^2 \rangle$.