Homework 13 solutions, Math 446, professor Agol, winter 2002

3.1.5.1 If p is homotopic to a point, show that p is the image of the boundary of a disk $\mathcal{D}$ which maps continuously into $\mathcal{C}$.

This problem should be stated for closed curves p, where the homotopy is through closed curves (with or without keeping the endpoint fixed).

If $p : [0,1] \to \mathcal{C}$ is a path, $p(0) = p(1)$, which is homotopic to a point (through closed paths), then there is a map $h : [0,1]^2 \to \mathcal{C}$ such that $h(t,1) = p(t), h(t,0) = c$, for some point $c \in \mathcal{C}$, and $h(0,s) = h(1,s)$, for all $s \in [0,1]$. Let $D = \{z \in \mathbb{C} | \ |z| \leq 1\}$, then we have a map $H : D \to \mathcal{C}$ given by $H(se^{2\pi it}) = h(s,t)$. This is well defined and continuous, since $H(se^{2\pi i0}) = H(se^{2\pi i1})$, and $H(0e^{2\pi it}) = H(0)$, for all $t \in [0,1]$.

3.2.2.1 Show that any closed path from the vertex of a bouquet of circles $e_1, e_2, ...$ is homotopic to a finite product of paths e_i^{+1} or e_i^{-1}.

Let B denote the bouquet of circles with vertex P, and let $p : [0,1] \to B$ be a closed path, such that $p(0) = p(1) = P$. Choose an open cover of B, where one set is $\mathcal{N}_{\frac{1}{2}}(P)$, which may also be described as the open set obtained by removing the midpoint m_i of each edge e_i, and $e_i - \{P\}$, forming an open cover $\mathcal{U}$ of B. Consider the open cover $\mathcal{V} = \{V \subset [0,1] \mid V \in \mathcal{H}(p^{-1}(U)),\ U \in \mathcal{U}\}$, that is the open cover of $[0,1]$ consisting of components of the preimages of open sets in $\mathcal{U}$. Then $\mathcal{V}$ has a finite subcover, so we may cover $[0,1]$ by finitely many intervals such that each interval maps to the interior of an edge, or maps to $\mathcal{N}_{\frac{1}{2}}(P)$. Let $V \in \mathcal{V}$, then $\overline{V} = [a,b] \subset [0,1]$. If $p(V) \subset e_i - \{P\}$, then $p(a) = p(b) = P$, since V is a component of $p^{-1}(e_i - P)$. So we have a map $p : [a,b] \to e_i$, and since e_i is an interval, we may homotope this to be straight. If $p(a), p(b)$ correspond to the "same" endpoint of e_i, then this will be homotoped to the constant map. If not, then it will be homotoped to a path representing $e_i^{\pm 1}$. Now, let $W = \cup\{V \in \mathcal{V} | V \subset p^{-1}(e_i - \{P\})\}$, then W consists of finitely many open intervals, so $[0,1] - W$ consists of finitely many closed intervals. Each such interval $[a,b]$ has the property that $p(a) = p(b) = P$, and $p([a,b]) \in \mathcal{N}_{\frac{1}{2}}(P)$. Since $\mathcal{N}_{\frac{1}{2}}(P)$ deformation retracts to P, we may homotope $p_{|[a,b]}$ rel endpoints using the deformation retract to be constant at P. Now, we may homotope p so that it is not constant on any interval, and then we have homotoped it to a path which is a finite product of $e_i^{\pm 1}$'s.

3.3.1.1 Suppose that there is a continuous map $\phi : D \to D$ of the disc into itself with no fixed points, that is $\phi(P) \neq P$ for each $P \in D$. Use the pair $\langle P, \phi(P) \rangle$ to define a point P' on the boundary circle S^1 such that $\rho(P) = P'$ is a retraction, thus proving the non-existence of ϕ.

Let P' be the intersection of the ray through $\phi(P)$ and P with S^1, *i.e.* $P' = \{(1-t)\phi(P) + tP | t > 0\} \cap S^1$. Then one may see that this map is continuous, since ϕ is continuous. For any $\epsilon > 0$, we may find a $\delta > 0$ so that if $|z - P| < \delta$, then $|\phi(z) - \phi(P)| < \epsilon$, by continuity. Let $\gamma = \min\{\epsilon, \delta\}$. Choose γ small enough that $B_\gamma(\phi(P)) \cap B_\gamma(P) = \emptyset$. Then there are two tangents to the circles of radius γ about $\phi(P)$ and P which separate one circle from the other, which meet at the point $\frac{1}{2}(P + \phi(P))$. The cone defined by these two tangents which contains P will intersect S^1 in an interval in S^1 about P'. Letting $\gamma \to o^+$, we see that we may make this cone intersect in arbitrarily small intervals about P' in S^1, and therefore $\rho(z)$ will lie in this interval when $|z - P| < \gamma$, so ρ is continuous. It is clear that if $P \in S^1$, then $\rho(P) = P$, so $\rho : D \to S^1$ is a retraction, a contradiction (as noted in the paragraph preceding this question).

3.3.2.1 Show that any vertex of a tree is a deformation retract of the whole tree, and hence that any graph has a bouquet of circles as a collapse. Deduce from Exercise 3.2.4.1 that the fundamental group of any graph is free.

Let v be a vertex of a tree $\mathcal{T}$. Choose a metric on the tree such that each edge has length 1. Then for any point $w \in \mathcal{T}$, there is a unique embedded interval $g_{wv} \subset \mathcal{T}$ with endpoints w and v.

Then let $d(w,v) = \text{length}(g_{wv})$. Define $h : \mathcal{T} \times [0,1] \to \mathcal{T}$ by $h(w) = w_t$, where by w_t we mean the unique point $w \in g_{wv}$ such that $d(w_t, v) = td(w,v)$. Thus $h(w,1) = w$, and $h(w,0) = v$, $h(v,t) = v$, and one may check that h is continuous (essentially because of the uniqueness of paths g_{wv}), so $\mathcal{T}$ deformation retracts to v.

Given a graph $\mathcal{G}$ with spanning tree $\mathcal{T}$, $\mathcal{G}$ collapses to a bouquet of circles by identifying $\mathcal{T}$ to a point, so $\pi_1(\mathcal{G})$ is a free group.

3.3.2.2 Use collapsing to show that the bouquet of two circles is a deformation retract of the perforated torus, Figure 140, and hence deduce that the fundamental group of the perforated torus is the free group of rank 2.

Since the torus may be obtained by identifying the opposite sides of the square $[0,1]^2$, and if we puncture the square $[0,1]^2 - B_\epsilon((0,0))$, $\epsilon < \frac{1}{2}$, then this deformation retracts to $\partial[0,1]^2$ by radial projection. When we identify opposite sides of $\partial[0,1]^2$, we get a bouquet of two circles. Since the deformation retract of a space has isomorphic fundamental group, we conclude that the punctured torus has free fundamental group on two generators.