Homework 14 solutions, Math 446, professor Agol, winter 2002

3.4.4.1 Give a criterion for a relator r_1 to yield a closed surface when the above construction is applied.

Suppose that we have a group presented by $\langle a_1, a_2, ..., a_n | r_1 \rangle$. We create a bouquet of n circles B with loops labelled by $a_1, ..., a_n$, and attach the boundary of a disk D corresponding to the relator r_1 to get a complex $\mathcal{C}$. In the interior of D, every point has a neighborhood homeomorphic to $\mathbb{R}^2$. For points in the interior of an edge of B, there must be exactly two points of ∂D attached to the point, so that in a neighborhood of the point, one has two half-spaces glued together to form an $\mathbb{R}^2$. So each generator must appear exactly twice in r_1 with exponent ± 1. To describe the condition needed near the vertex of B, take ∂D to be divided into $2n$ oriented intervals, labelled by generators a_i, each occurring twice in ∂D, corresponding to the word r_1. Then glue these in pairs, to obtain a surface. This surface will correspond to our complex obtained by attaching D to B iff it has one vertex, otherwise $\mathcal{C}$ would be obtained from this surface by identifying the vertices, which would not yield a surface. For example, the relator $a_1 a_2 a_3 a_1^{-1} a_2^{-1} a_3^{-1}$ does not give a surface.

3.4.4.2 Show that $\pi_1(S^2) = \{1\}$. Generalize the argument to S^n $(n \geq 2)$.

$S^n = \{(x_0, ..., x_n) \in \mathbb{R}^{n+1} \mid x_0^2 + \cdots + x_n^2 = 1\}$. Let $U_\pm = \{(x_0, ..., x_n) \in S^n \mid \pm x_0 > -1\}$. Then $U_\pm \cong \mathbb{R}^n$, *e.g.* by stereographic projection $(x_0, ..., x_n) \to \frac{1}{1 \pm x_0}(x_1, ..., x_n)$. $U_+ \cap U_- \cong \mathbb{R}^n - \{0\}$, so $U_+ \cap U_-$ is connected, since $n > 1$. So $\pi_1(U_+ \cap U_-) \to \pi_1(U_\pm)$ is trivial, and we see that $\pi_1(S^n)$ is trivial since it is a pushout of trivial groups.

3.4.5.2 Let $G_i = \langle a_{i1}, a_{i2}, ...; r_{i1}, r_{i2}, ... \rangle$ be realized by a surface complex $\mathcal{A}_i$. Let $\mathcal{C}$ be the complex formed by attaching each $\mathcal{A}_i$ by an edge e_i to a new vertex P (Figure 150). Prove a special case of an infinite Seifert-Van Kampen theorem to show that

$$\pi_1(\mathcal{C}) = \langle a_{11}, a_{12}, ..., a_{21}, a_{22}, ..., ...; r_{11}, r_{12}, ..., r_{21}, r_{22}, ..., ... \rangle.$$

Choose an open cover of $\mathcal{C}$ by taking the components of $\mathcal{C} - \{P\}$, along with $\mathcal{N}_1(P)$. Each component deformation retracts to $\mathcal{A}_i$ for some i, or to P. Using a covering argument as in problem 3.2.2.1, we see that any path may be written as a finite product of paths in finitely many $\mathcal{A}_i$'s. To see that the product is free, we take a loop which maps trivially, and a map of a disk with boundary mapping to the loop. By the covering argument, this may be homotoped into finitely many $\mathcal{A}_i$'s. Then one may inductively use Seifert-Van Kampen on these finitely many $\mathcal{A}_i$'s to show that this loop may be written as a product of conjugates of relators in the $\mathcal{A}_i$'s, proving that the presentation gives the group.

3.5.2.1 Show that $G \times H$ results from $G * H$ by adding relations $gh = hg$ for each generator $g \in G, h \in H$.

If we add the relations $gh = hg$ to the group $G * H$, for all generators $g \in G, h \in H$, then this group maps to $G \times H$, by sending each element of $g \in G$ to $(g, 1)$, and each element of $h \in H$ to $(1, h)$, and extending by the universal property of $G * H$. Since $(g, 1)(1, h) = (1, h)(g, 1)$, this will give a well-defined homomorphism. Conversely, we may map $(g, h) \in G \times H$ to $gh \in G * H / \{gh = hg\}$. This is a homomorphism, since if we take $(g_1, h_1) \cdot (g_2, h_2) \mapsto g_1 h_1 g_2 h_2$, we may use the relators to commute h_1 past g_2 one generator at a time to get $g_1 g_2 h_1 h_2$, which comes from $(g_1 g_2, h_1 h_2)$. Also, this map is onto, since we may use the relators to write any word in $G * H / \{gh = hg\}$ in the form gh. Thus, the two groups are isomorphic.