Homework 6, Math 446, professor Agol, winter 2002

1. Show that $\mathbb{R}^2_+ = \{(x,y) \in \mathbb{R}^2 \mid y \geq 0\} \not\cong \mathbb{R}^2$. Deduce that the boundary of a bounded manifold is topologically invariant.

As shown in 0.3.7, no simple arc a embedded in $\mathbb{R}^2$ separates $\mathbb{R}^2$, *i.e.* $\mathbb{R}^2 - a$ has only one component. Let $b = \{(x,y) \in \mathbb{R}^2_+ | \; x^2+y^2 = 1\} \subset \mathbb{R}^2_+$, then b is a simple arc in $\mathbb{R}^2_+$, since for example, projection onto the first coordinate gives a homeomorphism with the interval $[-1,1]$. Also, $\mathbb{R}^2_+ - b$ has two components, since the map $r : \mathbb{R}^2_+ \to [0,\infty)$, given by $r(x,y) = x^2+y^2$ is continuous. $b = r^{-1}(1)$, so $r^{-1}([0,1))$ and $r^{-1}((1,\infty))$ are the distinct components of $\mathbb{R}^2_+ - b$. If we had a homeomorphism $h : \mathbb{R}^2_+ \to \mathbb{R}^2$, then $h_{|b}$ would give an embedded interval $h(b)$ in $\mathbb{R}^2$. But then $\mathbb{R}^2 - b$ would have two components $h(r^{-1}([0,1)))$ and $h(r^{-1}((1,\infty)))$, a contradiction. Thus, there is no homeomorphism between $\mathbb{R}^2$ and $\mathbb{R}^2_+$.

Now, let $x \in M - int(M) = \partial M$. Then x has an open neighborhood $U \cong \mathbb{R}^2_+$. We want to show that $\partial \mathbb{R}^2_+ = \mathbb{R} \times \{0\}$. If $(x,y) \in \mathbb{R}^2_+ - \mathbb{R}\times\{0\}$, then the open ball of radius y about (x,y) gives neighborhood homeomorphic to $\mathbb{R}^2$. If we have a point $(x,0)$ which has a neighborhood $U \cong \mathbb{R}^2$ then $\exists \epsilon > 0$ such that $B_\epsilon((x,0)) \cap \mathbb{R}^2_+ \subset U$, since $\mathbb{R}^2_+$ has the subspace topology in $\mathbb{R}^2$. But the arc $\{(x,y) | x^2+y^2 = (\epsilon/2)^2\}$ is an arc which separates $\mathbb{R}^2_+$, and thus separates U. But then U would have an embedded arc which separated it, a contradiction. So we see that $x \in \partial M$ iff x has a neighborhood $U \cong \mathbb{R}^2_+$ such that $x \in \mathbb{R}\times\{0\}$. Since this is a purely local characterization of ∂M, we see that ∂M is a topological invariant: if $h : N \to M$, is a homeomorphism, then $h|_{\partial N}$ is a homeomorphism of ∂N with ∂M. The definition of $\partial M = M - int(M)$ is clearly topologically invariant, but the point is that we can characterize ∂M as the points $x \in M$ with a neighborhood $U \cong \mathbb{R}^2_+$ such that $x \in \partial U$, which is the intuitive notion that we had for the boundary of a manifold. This also shows that the boundary of a 2-manifold is a 1-manifold, and this shows that the boundary is non-empty in any example where a point has an open neighborhood $\cong \mathbb{R}^2_+$.

2. Suppose we have a polygonal arc $p : [0,1] \to \mathbb{R}^2$ which connects distinct points $a, b \in \mathbb{R}^2$. Prove that there is a subset of $p([0,1])$ which is an embedded polygonal arc connecting a and b.

We will assume that we have a path $p : [0,x] \to \mathbb{R}^2$. Since p is polygonal, there is a sequence $0 = a_0 < a_1 < a_2 < \cdots < a_n = x$ such that p is affine on the interval $[a_i, a_{i+1}]$, $0 \leq i < n$. Thus, $p([a_i,a_{i+1}])$ is an embedded line segment between $p(a_i)$ and $p(a_{i+1})$ for all $0 \leq i < n$. We will induct on the number of line segments n. If p is not $1-1$, then there are indices $0 \leq i \leq j < n$ and points $0 \leq c < d \leq 1$ such that $a_i \leq c \leq a_{i+1}$, and $a_j \leq d \leq a_{j+1}$, and $p(c) = p(d)$. We may assume $i < j$ since if $i = j$, then $c = d$, since p is $1-1$ on the interval $[a_i, a_{i+1}]$. Suppose $j = i+1$. Then $p(a_{i+1}) = p(a_j)$, and $p(c) = p(d)$. Since p is $1-1$ on each interval $[a_i, a_{i+1}]$ and $[a_{i+1}, a_{i+2}]$ we see that $c, d \neq a_{i+1}$. So we have two line segments agreeing at two distinct points, which means that they must lie on the same line. We must have either $p([a_i,a_{i+1}]) \subset p([a_{i+1},a_{i+2}])$ or $p([a_i,a_{i+1}]) \supset p([a_{i+1},a_{i+2}])$. In the first case, we can take $c = a_i$. In the second case, we may take $d = a_{i+2}$. If $i+1 < j$, then we remove the segment $[c,d]$, getting a path $p' : [0, x-(d-c)] \to \mathbb{R}^2$, such that

$$p'(t) = \begin{cases} p(t) & 0 \leq t \leq c \\ p(t+(d-c)) & c \leq t \leq x-(d-c) \end{cases} \tag{1}$$

The path p' will have at most $n-1$ segments, since we have removed at least one segment, and we replace p with p', x by $x-(d-c)$. Thus, by induction, this process will terminate with an embedded polygonal path connecting a and b, with image lying in $p([0,1])$ for our original path p.

3. If we have a Jordan curve P, then by the Jordan curve theorem, there is a unique bounded component D of $\mathbb{R}^2 - P$, so that $\Delta = D \cup P$ is a compact set. Suppose that Δ is not convex. Take the convex hull $C(P)$ of P, that is, take the intersection over all half-spaces containing P.

Since Δ is not convex, $C(P) \neq \Delta$, otherwise, any two points in Δ would be connected by an arc in $C(P) = \Delta$, so Δ would be convex. The boundary $\partial C(P)$ is a Jordan curve, as we showed in class, and there is a point $p \in \partial C(P) - P$. Consider the visual angle of P from p. If it is less than π, then we could find a line l closer to P separating p from P, so that $p \notin C(P)$, a contradiction. Thus, the visual angle of P from p is π, and therefore there is a line through p intersecting P in two disjoint sets.

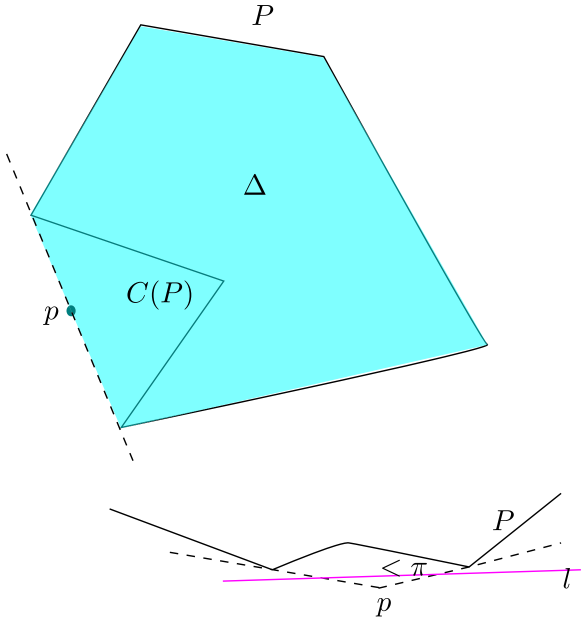

Figure 1: The convex hull of a Jordan curve.