Homework 7 solutions, Math 446, professor Agol, winter 2002

1. Prove Sperner's lemma for 4-colorings of the vertices of a subdivision S of the 3-simplex (tetrahedron) T: if one colors the vertices of T four distinct colors, and any vertex of S is colored one of the colors of the vertices of the face of T in which it lies, then there will be one tetrahedron in S with four distinct colored vertices. Use this to prove Brouwer's fixed point theorem for the 3-ball B^3. Discuss generalizations to higher dimensions. Can you show that the antipodal map $-id : S^n \to S^n$ is not homotopic to the identity using Brouwer's fixed point theorem for B^n?

Since S is a subdivision of T, we know that S is obtained from T by a sequence of elementary subdivisions $T = T_0, T_1, ..., T_n = S$. So T_{i+1} is obtained from T_i by adding a new vertex to $vert(T_i)$ and performing the corresponding elementary subdivision. By induction, assume that for any 4-coloring of $vert(T_i)$, there is a tetrahedron σ of T_i with vertices colored four different colors. When we add a point x to T_i, then we perform the corresponding elementary subdivision of each simplex of T_i containing x to get the subdivision T_{i+1}. Suppose we have a 4-coloring of $vert(T_{i+1})$. Then it is clear that this induces a 4-coloring on $vert(T_i)$ satisfying the condition that any vertex is colored one of the colors of the corner vertices of T . Let σ be a simplex of T_i with four colors. If $x \notin \sigma$, then σ will be a simplex of T_{i+1}, so we are done. Otherwise, $x \in \sigma$. Then x will lie in face $\sigma' \subset \sigma$, and will be colored the same color as a vertex v of σ'. Then the simplex with vertices $vert(\sigma) - \{v\} \cup \{x\}$ will have its vertices distinctly colored. By induction on elementary subdivisions, we have shown that S contains a tetrahedron with vertices colored 4 colors (clearly the argument generalizes to arbitrary dimensions).

Now, take a regular tetrahedron T. We may consider $T \subset \mathbb{R}^4$, as the convex hull of the four elementary vectors $\{(1, 0, 0, 0), ..., (0, 0, 0, 1)\} \in \mathbb{R}^4$. Take the barycenter $b = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4})$, and consider the map $\partial T \to S^2$ given by $x \mapsto \frac{x-b}{\|x-b\|}$. Then each 2-dimensional face of ∂T maps to a region of S^2, decomposing S^2 into four regions, such that every point on S^2 is contained in at most 3 regions. Suppose we have a continuous map $f : T \to T$. Suppose that f has no fixed point. Then the map $F : T \to S^2$ defined by $F(x) = \frac{f(x)-x}{\|f(x)-x\|}$ is continuous. Color each point of T by the region of S^2 it lies in, resolving ties arbitrarily. From the way that we chose the coloring, it follows that the vertices are colored four different colors, and each point in a face of T will be colored the same as one of the vertices of the face (if not, then $f(x) - x$ would be pointing towards a face that it lies in, which is impossible). So the coloring of any subdivision of T will satisfy the hypotheses of Sperner's lemma. Then for small enough ϵ, every point in S^2 in a ball of radius ϵ lies in at most 3 regions of S^2. Choose δ so that $F(B_\delta(x)) \subset B_\epsilon(f(x))$ for all $x \in T$, which we may do since T is compact so F is uniformly continuous. By a previous homework exercise, the nth barycentric subdivision of T will have mesh arbitrarily small (or take some other sequence of subdivisions with this property). If the mesh is small enough that each tetrahedron lies in a ball of radius δ, then each point inside a δ ball will be colored at most 3 different colors, so each tetrahedron in the triangulation will have at most 3 different colors, contradicting our generalized Sperner's lemma. Thus, every map $f : T \to T$ has a fixed point. Since $T \cong B^3$, the same holds for the 3-ball. The same argument works in n dimensions, with 2 replaced by $n - 1$ and T replaced by an n-simplex.

Suppose that the map $-id : S^n \to S^n$ is homotopically trivial. Then we have a homotopy $f : S^n \times [0, 1] \to S^n$ such that $f(x, 1) = -x$, and $f(x, 0) = c$, for some constant $c \in S^n$. Then we can extend $-id$ to a map $F : B^{n+1} \to S^n \subset B^{n+1}$ by $F(rx) = f(x, r)$. F is well-defined since $F(0x) = F(0y) = c$, for any $x, y \in S^n$. F is continuous, since f is continuous, and $B^{n+1} = S^n \times [0, 1]/\{(x, 0) \sim (y, 0)\}$ under the quotient topology, and $F \circ p = f$, where $p : S^n \times [0, 1] \to B^{n+1}$ is the continuous map $p((x, r)) = xr$. Thus, we have a map $\iota \circ F : B^{n+1} \to B^{n+1}$, where $\iota : S^n \to B^{n+1}$, and $\iota \circ F$ must have a fixed point x. But $\iota \circ F(B^{n+1}) \subset S^n$, so $\iota \circ F(x) \in S^n$, so $\iota \circ F(x) = \iota \circ (-id)(x) = -x \neq x$, a contradiction. Thus, f must have had a fixed point.

2. A topological space T has the fixed point property if for any map $f : T \to T$, f has a fixed point, *i.e.* a point x in T such that $f(x) = x$. Which of the following has the fixed point property?

a. (solid) square

This has the fixed point property since it is homeomorphic to B^2, and the fixed point property is a topological property.

b. circle

$x \mapsto -x$ does not have a fixed point.

c. 2-sphere

ditto.

d. $\mathbb{R}^2$

$(x, y) \mapsto (x + 1, y)$ does not have a fixed point.

e. $T^2 = S^1 \times S^1$

$(x, y) \mapsto (-x, -y)$ does not have a fixed point, for example.

f. The letter T (thought of as a graph).

This has the fixed point property. Let $\mathsf{T} = a \cup b \cup c$ be the union of three intervals in the obvious way, $a \cap b \cap c = x$, a single point. Suppose we have a continuous map $f : \mathsf{T} \to \mathsf{T}$ without a fixed point. Then $f(x) \in \mathsf{T} - \{x\}$, and thus $f(x) \in a - \{x\}$ or $f(x) \in b - \{x\}$ or $f(x) \in c - \{x\}$. By relabelling, we may assume $f(x) \in a - \{x\}$. If $f(a) \subset a$, then f restricts to a map on a, so it would have a fixed point, since $a = B^1$ has the fixed point property. Thus, $f(a) \not\subset a$. We must then have $f^{-1}(x) \cap a \neq \emptyset$, since then $f(a - \{x\}) \subset a - \{x\}$, since it would have to map $a - \{x\}$ to a component of $\mathsf{T} - \{x\}$. Consider the largest interval $[x, y] \subset a$ such that $f([x, y]) \subset a$, and therefore $f(y) = x$. If we consider the interval a parametrized so that for $z \in a$, $x \leq z \leq w$, where w is the other endpoint of a, then we have $f(x) > x$, $f(y) = x < y$, so by the intermediate value theorem there must be a point $x < z < y$ such that $f(z) = z$. Thus f has a fixed point.