Homework 8 solutions, Math 446, professor Agol, winter 2002

1. Prove that the group presented by $\langle a, b \mid abA = b^2, baB = a^2\rangle = 1$.

Multiplying the first relation by B on both sides on the right, we get $abAB = b$. Inverting both sides of the second relation, we get $bAB = A^2$, then multiply both sides on the left by a to get $abAB = A = b$. Now, substitute $b = A$ into the first relation to get $aAA = A^2$, and cancelling, we get $a = 1$, so $b = A = 1$, and the group is trivial.

2. Prove that the Burnside group $B(n, 2) = \langle g_1, ..., g_n \mid g^2 = 1, \forall g \in \langle g_1, ..., g_n\rangle\ \rangle$ is finite.

In fact, we may show that the group is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^n$. Since $g_i^2 = 1$, we have $g_i^{-1} = g_i$, and therefore $(g_ig_j)^2 = g_ig_jg_ig_j = 1$ implies $g_ig_j = g_j^{-1}g_i^{-1} = g_jg_i$, so every pair of generators commute. Since $\langle g_i|g_i^2\rangle = \mathbb{Z}/2\mathbb{Z}$, we have a direct product of n $\mathbb{Z}/2\mathbb{Z}$'s.

0.5.7.3 Sketch the Cayley diagram for the free group $F_2 = \langle a, b\rangle$.

See Cayley diagram of free group on 2 generators

0.5.7.4 Describe the Cayley diagrams of the *free abelian groups* $\mathbb{Z}^m = \langle a_1, ..., a_m|a_ia_j = a_ja_i\rangle$ as figures in $\mathbb{R}^m$.

The Cayley graph may be embedded as a lattice, with points $\{(z_1, ..., z_m)|z_i \in \mathbb{Z}\}$, and oriented lines connecting pairs of points of the form $(z_1, ..., z_k, ..., z_m)$ and $(z_1, ..., z_k + 1, ..., z_m)$. Essentially, we'll see m-hypercubes stacked infinitely in every direction.

0.5.7.5 Figure 46 shows the Cayley diagram of a group. Why is this group non-abelian? Show that the group is the group of symmetries of an equilateral triangle.

If we start at the top vertex, and follow the edges a, then b, we end up in the lower left of the diagram. But if we follow b, then a, we end up in the lower right. So a and b do not commute.

If we take an equilateral triangle, then let b be rotation by $2\pi/3$, and let a be flipping about some median of the triangle. Then a has order two, and b has order three, just like the group of the Cayley graph. Also, conjugating rotation by $2\pi/3$ by flipping, *i.e.* aba gives rotation by $-2\pi/3$, or b^{-1}, just like the group. Given these relations, we may construct the Cayley graph of this group of symmetries, and see that it is that given in figure 46.

0.5.8.1 Show that $\langle a, b|abaB\rangle = \langle c, d|c^2d^2\rangle$.

Define $\phi : \langle a, b|abaB\rangle \to \langle c, d|c^2d^2\rangle$ by $\phi(a) = cd, \phi(b) = d$. Then we see that $\phi(abaB) = \phi(a)\phi(b)\phi(a)\phi(b)^{-1} = cddcdD = C(c^2d^2)c = 1$. Thus, ϕ is well-defined. We may define ψ by $\psi(c) = aB$, and $\psi(d) = b$. Then $\psi(c^2d^2) = aBaBb^2 = aBab = aB(abaB)bA = 1$, so ψ is well-defined, and we see that $\phi \circ \psi(c) = \phi(aB) = cdd^{-1} = c$, $\phi \circ \psi(d) = \phi(b) = d$. Similarly, $\psi \circ \phi(a) = \psi(cd) = aBb = a$, and $\psi \circ \phi(b) = \psi(d) = b$, so ϕ and ψ are inverse to each other, and are therefore isomorphisms.