Homework 3, Math 446, professor Agol, winter 2002

0.1.9.1. Show that any homeomorphism of $\mathbb{R}^1$ is isotopic either to the identity or -identity.

From Problem 2, Math 445 midterm 2, we know that a homeomorphism $f : \mathbb{R} \to \mathbb{R}$ is monotonic - either increasing or decreasing. If f is increasing, let $\epsilon = 1$, if f is decreasing, let $\epsilon = -1$. Consider the homotopy $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ defined by $F_t(x) = tf(x) + (1-t)\epsilon x$. If f is increasing, we have for $x < y$, $f(x) < f(y)$, so $F_t(x) = tf(x) + (1-t)x < tf(y) + (1-t)y$, so F_t is increasing and continuous for all t, and $F_0 = id_{\mathbb{R}}$, $F_1 = f$. Assuming f is increasing, F_t is onto for $t \neq 0$, since if $y = f^{-1}(0)$, then for $x > y$, we have $F_t(x) = tf(x) + (1-t)x > tf(y) + (1-t)x = (1-t)x$. So $\sup F_t(x) = \infty$. Similarly, $\inf F_t(x) = -\infty$. Since $F_t(\mathbb{R})$ is connected, it is an interval, so it must be $\mathbb{R}$ since it is unbounded above and below. Thus, F_t is a homeomorphism for all t. If f is decreasing, then ϵF_t is a homeomorphism for all t, so F_t is too. Thus, F is an isotopy of f to $\epsilon id_{\mathbb{R}}$.

0.2.1.1. Show that an infinite simplicial complex is not compact.

Let Σ be an infinite simplicial complex, with vertex set V. Since Σ is infinite, V must be infinite, otherwise it would be a subcomplex of $\Delta^{\#V-1}$. For each vertex $v \in V$, let $st(v)$ be the open star of v. If $\sigma = v, v_1, ..., v_n$ is an n-simplex containing v, then $st(v) \cap \sigma = \sigma - v_1, ..., v_n$. This is an open subset of the simplex σ in its subspace topology, since it is defined by the condition $\alpha_0 > 0$, where $\sigma = \alpha_0 v + \alpha_1 v_1 + ... + \alpha_n v_n | 0 \leq \alpha_i, \sum \alpha_i = 1$. Since the topology of Σ is the quotient topology of its simplices under identification, $st(v)$ is open iff its intersection with each simplex is open in the subspace topology on the simplex. Thus, $st(v)$ is open. This forms an open cover of Σ, since any point in Σ is either a vertex, or is in the interior of a simplex σ, so it has some affine coefficient $\alpha_i > 0$, which means it is in $st(v_i)$ for that vertex. Since $st(v) \cap V = \{v\}$, if the open cover $\{st(v) | v \in V\}$ had a finite subcover $\{st(v) | v \in W\}$, $\#W < \infty$, then for any $u \in V - W$, we have $w \notin \{st(v) | v \in W\}$, a contradiction. Thus, Σ is not compact.

0.2.1.3. In a simplicial n-manifold, show that the faces not containing P in the neighborhood star of P constitute a topological S^{n-1}.

This problem is incorrect, in that Stillwell has not defined what a simplicial n-manifold is. There are simplicial complexes homeomorphic to an n-manifold which do not satisfy this property. Usually, a simplicial n-manifold is defined inductively by the requirement that the boundary of the star of a vertex is a simplicial $n-1$-sphere, in which case this problem is correct by definition.

0.2.4.1. The first is simplicial. The second is not, since two triangles intersect in a disjoint union of an edge and a vertex.

0.2.4.2. The barycentric subdivision is obtained by adding, for each simplex $\sigma = \{v_0, ..., v_n\}$, a new vertex $v_\sigma = (v_0 + ... + v_n)/(n+1)$, the barycenter in affine coordinates on the simplex. Then we do elementary subdivisions on the highest dimension simplices first (assuming the complex is locally finite dimensional), that is we replace σ with $n+1$ simplices $\{v_0, ..., v_n, v_\sigma\} - \{v_i\}$, $i = 0, ..., n$, and inductively do the same for lower dimensional simplices. This gives the barycenter construction as a subdivision. By convexity, the diameter of a simplex is the maximal distance between its vertices. A direct computation shows that the length of each of the edges of the simplices in the barycentric subdivision is at most $\frac{n}{n+1}\, diam(\sigma)$. Thus, the diameter goes to zero when we barycentric subdivide more and more.

0.2.4.4. Suppose Σ is a simplicial complex which is a simplicial n-manifold with boundary. Then every $n-1$-simplex of $|\Sigma|$ lies in at most two n-simplices of Σ. The boundary are the points which lie in an $n-1$-simplex which is in only one n-simplex (this property should be proved by induction on dimension, and use the fact that the link of a vertex in an n-manifold with boundary is either an $n-1$-sphere or ball, depending on whether the vertex lies in the interior or boundary of the n-manifold). When we subdivide, this property will be preserved, so the subdivision preserves the boundary.

Given a simplex $\sigma = \{v_0, ..., v_n\}$, and a point $Q \in \sigma$, we may perform an elementary subdivision. If σ has orientation $\epsilon[v_0, ..., v_n]$, $\epsilon = \pm 1$, then if $\{v_0, ..., v_n, Q\} - \{v_j\}$ is affinely independent, the induced orientation on this simplex is obtained by replacing v_j with Q in the linear order. One may check that this gives an orientation of the subdivided simplex, and induces a consistent orientation on each subdivided $n-1$-face. Essentially, if $\{v_0, ..., v_n, Q\} - \{v_j\}$ and $\{v_0, ..., v_n, Q\} - \{v_k\}$ intersect in the face $\{v_0, ..., v_n, Q\} - \{v_j, v_k\}$, where $k < j$, then if we replace v_k by Q in the linear ordering, and remove v_j, we get $\epsilon(-1)^j[v_0, ..., v_{k-1}, Q, ..., \hat{v_j}, ..., v_n]$. But if we replace v_j by Q, then remove v_k, we get the linear order $\epsilon(-1)^k[v_0, ..., v_{k-1}, \hat{v_k}, ..., v_{j-1}, Q, ..., v_n]$. We must do $j - k - 1$ transpositions to take one order to the other, so the two orientations differ by -1. Thus, we get a consistent orientation on the simplex. Also, the induced orientations on the $n-1$ faces not containing Q will be the same as that of σ, since we remove a vertex from the same spot in the ordering. Thus, if the simplicial complex is orientable, then the subdivision will be as well. One may check that the converse holds too. So the complex is orientable iff its elementary subdivision is. Since a subdivision is obtained by repeated elementary subdivisions, we see that the orientability of the complex is invariant under subdivisions.