Homework 5, Math 446, professor Agol, winter 2002

Prove that two graphs are homeomorphic if and only if they have a common subdivision.

Let Γ_1 and Γ_2 be graphs, *i.e.* abstract simplicial 1-complexes. Let $f : |\Gamma_1| \to |\Gamma_2|$ be a homeomorphism between the geometric realizations of the graphs. Let $V_i \subset |\Gamma_i|$ be the vertex set of Γ_i. Then let $U_1 = V_1 \cup f^{-1}(V_2)$ and $U_2 = f(V_1) \cup V_2$. We obtain canonical subdivisions Γ_i' of Γ_i such that Γ_i' has vertex set U_i. To see this, we may perform elementary subdivisions of Γ_i by adding in vertices of $U_i - V_i$ one at a time. Then, we can check that the order in which we perform these elementary subdivisions does not affect the resulting subdivision. This boils down to checking that for two points $u, v \in \Delta^1$, performing elementary subdivisions on u then v gives the same complex as performing elementary subdivisions on v then u. Since Γ_i' is a subdivision of Γ_i, $|\Gamma_i'| \cong |\Gamma_i|$ by the inclusion map. So we have a map $f' : |\Gamma_1'| \to |\Gamma_2'|$, which takes vertices of Γ_1' to vertices of Γ_2'. Now, we want to show that Γ_1' and Γ_2' are isomorphic simplicial 1-complexes. Since $\Gamma_i' - U_i$ is a collection of interiors of 1-simplices of Γ_i', f' restricts to a homeomorphism of $\Gamma_1' - U_1$ to $\Gamma_2' - U_2$. If we have a 1-simplex $|\{u, v\}| \in |\Gamma_1'|$, then $f'(|\{u, v\}|) = |\{f'(u), f'(v)\}|$, since a 1-simplex is determined by its vertices, so the interior of $|\{u, v\}|$ must map to the component of $\Gamma_2' - U_2$ which has endpoints $f'(u), f'(v)$, by continuity. Thus, Γ_i' are isomorphic simplicial 1-complexes.

0.2.4.3. A 1-dimensional cell complex is obtained by taking a collection of intervals, and identifying endpoints of these intervals in some fashion. It is not necessarily a cell complex, because an interval may have its endpoints identified with each other, or two intervals may have their endpoints identified. The first barycentric subdivision adds a new vertex to the interior of each edge, so any new 1-cell will not have its endpoints identified. If two 1-cells share both endpoints (but these endpoints are distinct), then the cells in the barycentric subdivision will intersect in at most one point, since if they intersect in a point, it came from the original points, in which case the other endpoints of the intervals correspond to distinct interior vertices, or they share a point coming from an interior point, in which case the endpoints are distinct since they correspond to the endpoints of the interval, which we assumed didn't form a loop. So the second subdivision gets rid of intervals which share common endpoints.

0.3.1.1. Let $\mathcal{O}'$ be the connected component of $\mathcal{O}$ containing P. Then $\mathcal{O}'$ is path connected, as we showed last semester. Given an arc $a : [0,1] \to \mathcal{O}'$ connecting P and Q in $\mathcal{O}'$, choose an open cover by open balls of $\mathcal{O}'$, and take a finite subcover τ of $a([0,1])$. Let $\mathcal{B}_1 \in \tau$ be a ball containing P. If $Q \in \mathcal{B}_1$, then we are done. Otherwise, by induction, suppose $\mathcal{B}_1, ..., \mathcal{B}_k \in \tau$ have been defined, such that $\mathcal{B}_i \cap \mathcal{B}_{i+1} \neq \emptyset$, and $Q \notin \cup_{i=1}^k \mathcal{B}_i$. Then $a^{-1}(\cup_{i=1}^k \mathcal{B}_i)$ is an open subset of $[0,1]$ containing 0 (since $a(0) = P$) and not containing 1 (since $a(1) = Q$). Since τ covers $a([0,1])$, let $\mathcal{B} \in \tau$ be an open ball such that $p = \sup a^{-1}(\cup_{i=1}^k \mathcal{B}_i) \in a^{-1}(\mathcal{B})$. We also have that $\mathcal{B} \cap (\cup_{i=1}^k \mathcal{B}_i) \neq \emptyset$, since p is a limit point of $a^{-1}(\cup_{i=1}^k \mathcal{B}_i)$, and $p \in a^{-1}(\mathcal{B})$, an open set. Choose $j \leq k$ such that $\mathcal{B}_j \cap \mathcal{B} \neq \emptyset$, and rename $\mathcal{B} = \mathcal{B}_{j+1}$, throwing away the other balls $\mathcal{B}_i, i > j$. If $Q \in \mathcal{B}_{j+1}$, then we are done, otherwise we continue. This process will terminate, since $\sup a^{-1}(\cup \mathcal{B}_i)$ is strictly increasing, and τ is a finite cover of $a([0,1])$, so we must eventually reach a ball containing Q. Thus, we get a chain of balls connecting P and Q, and we see that $\mathcal{O}_P \subset \mathcal{O}'$. As shown in the proposition, any point Q in $\mathcal{O}_P$ is connected by a polygonal arc to P, so is in $\mathcal{O}'$. This shows the stated equalities of the exercise.