Definitions of simplicial complexes, Math 446, professor Agol, winter 2002

Simplicial complexes will be one of the main classes of topological spaces we will study this semester.

Let V be a finite set. An **abstract simplicial complex** or **schema** $K \subset 2^V = \mathcal{P}(V)$ is a family of subsets of V, called **simplexes**, such that

1. if $v \in V$, then $\{v\} \in K$;
2. if $s \in K$ and $s' \subset s$, then $s' \in K$.

Remark: Stillwell requires that a simplicial complex K is locally finite, *i.e.* for each simplex $s \in K$, there are only finitely many $\tau \in K$ such that $s \in \tau$. We won't assume this in the definition, although in practice most simplicial complexes we consider will have this property.

We call V the **vertex set** or **vertices** of K and sometimes denote it by $V = \mathrm{Vert(K)}$. A simplex $s \in K$ having $q+1$ vertices is a **q-simplex**.

If K and L are abstract simplicial complexes, then a **simplicial map** $\phi : K \to L$ is a function $\phi : \mathrm{Vert(K)} \to \mathrm{Vert(L)}$ such that if $s \in K$, then $\phi(s) \in L$. K and L are **isomorphic** simplicial complexes if there is a simplicial map $\phi : K \to L$ which is bijective.

Remark: Abstract simplicial complexes and simplicial maps form a category.

Let K be an abstract simplicial complex with $V = \mathrm{Vert(K)}$. Define $I^{(V)}$ to be the set of all functions $\phi : V \to I$ that are zero for all but finitely many $v \in V$. For each $v \in V$, define $\tilde{v} \in I^{(V)}$ by $\tilde{v}(v) = 1$ and if $u \neq v$, then $\tilde{v}(u) = 0$. If $\sigma = \{v_0, ..., v_n\} \in K$ is an n-simplex, define $\tilde{\sigma} \in I^{(V)}$ as the family of all convex combinations of $\{\tilde{v_0}, ..., \tilde{v_n}\}$, *i.e.* $\tilde{\sigma} = \{\alpha_0\tilde{v_0} + \cdots + \alpha_n\tilde{v_n} \mid \sum \alpha_i = 1, a_i \geq 0\}$. Then $\tilde{\sigma}$ is an affine n-simplex in $\{\tau \in I^{(V)} | \tau(x) > 0 \Rightarrow x \in \sigma\} \cong I^{\sigma} \subset \mathbb{R}^{\sigma}$. Define $|K| = \cup_{\sigma \in K}\tilde{\sigma}$, with the quotient topology from the simplices $\tilde{\sigma}$. That is, given the topology on $\tilde{\sigma}$ as a simplex $\cong \Delta^n$, we choose the weakest topology on $|K|$ so that the inclusion $\tilde{\sigma} \subset |K|$ is continuous. We call $|K|$ the **geometric realization** of K. The **simplicial complex** is the collection $\tilde{K} = \{\tilde{\sigma} | \sigma \in K\}$. We want to distinguish this from the topological space $|K|$ since we want to keep track of all the simplices and their affine structure, as well as the underlying topological space.

A topological space is a **polyhedron** if there exists an abstract simplicial complex K and a homeomorphism $h : |K| \to X$. The ordered pair (K, h) is called a **triangulation** of X.

The **n-skeleton** of an abstract simplicial complex K is the subset $K^{(n)} = \{\sigma \in K | \#\sigma \leq n+1\}$, where $\#$ denotes cardinality. Then the n-skeleton of $\tilde{K}$ is $\widetilde{K^{(n)}}$. We will many times confuse an abstract simplicial complex with the complex or its geometric realization. A **face** σ' of an abstract simplex σ is a subset $\sigma' \in \sigma$, and we also call $\tilde{\sigma'}$ a face of $\tilde{\sigma}$. The **dimension** of K is the largest n such that K has a simplex of dimension n, in which case we call K an n-complex.

For an abstract simplex σ of cardinality $n+1$, let $[n+1] = \{1, ..., n+1\}$. Orderings $\mathcal{O}(\sigma)$ of σ are another name for the set of bijective maps from $[n+1]$ to σ. If $\sigma = \{v_0, v_1, ..., v_n\}$, we will denote an ordering by $[v_0, v_1, ..., v_n]$. Two orderings are equivalent if they differ by an even permutation: $\phi_i : [n+1] \to \sigma$, $i = 1, 2$, $\phi_1^{-1} \circ \phi_2 : [n+1] \to [n+1]$ is an even permutation of $[n+1]$, *i.e.* it is in the alternating subgroup of the symmetric group, so it is the composition of an even number of transpositions. An orientation of an abstract simplex σ is a map $o : \mathcal{O}(\sigma) \to \{\pm 1\}$ which changes sign under a transposition of a pair of elements in the linear order. There are only two orientations on a simplex, o and $-o$, and we will denote the orientation o by a linear order $[v_0, v_1, ..., v_n]$ such that $o([v_0, v_1, ..., v_n]) = 1$. If we take an $n-1$-simplex $\sigma' = \{v_0, .., \hat{v_i}, ..., v_n\}$ face of an n-simplex $\sigma = \{v_0, ..., v_n\}$ with orientation $[v_0, ..., v_n]$, then the induced orientation on σ' is $(-1)^i[v_0, ..., \hat{v_i}, ..., v_n]$. One may check that this does not depend on the particular linear order we have chosen. We may do i transpositions to put v_i in the first spot of $[v_0, ..., v_n]$, to express the orientation on σ as $(-1)^i[v_i, v_0, ..., \hat{v_i}, ..., v_n]$. Then if we remove v_i, we get an orientation $(-1)^i[v_0, ..., \hat{v_i}, ..., v_n]$ on σ'. Any even permutations not involving v_i will not affect the orientations of σ or σ'. So the induced orientation is independent of the choice of linear order.

An **orientation** on an abstract simplicial is an assignment of orientations to all the n-simplices of K such that for each $n-1$-simplex τ, if σ and σ' are n-simplexes containing τ, then the induced orientations of τ from σ and σ' are opposite. Notice that this implies that the number of n-simplices containing τ is at most 2, since τ has only two orientations.

An **elementary subdivision** of an abstract simplicial complex K at the simplex $\sigma \in K$ is the simplicial complex K_σ with vertex set $\mathrm{Vert(K)} \cup \mathrm{Q}$ (where we assume $Q \notin \mathrm{Vert(K)}$). The simplices of K_σ consist of simplices $\sigma' \in K$ such that $\sigma \notin \sigma'$ and, for simplices σ' such that $\sigma \subset \sigma'$, we remove each vertex of σ and add Q: $K_\sigma = \cup_{v \in \mathrm{Vert}(\sigma)} \{\sigma' \cup \{Q\} - \{v\} \mid \sigma \in \sigma'\} \cup \{\sigma' \in K \mid \sigma \notin \sigma'\}$. It is easy to check that $|K| \cong |K_\sigma|$. A **subdivision** L of a complex K is obtained by a sequence of elementary subdivisions $K_0 = K, ..., K_m = L$ such that K_{i+1} is obtained from K_i by elementary subdivision. Two simplicial complexes are **P.L. equivalent** if they have common subdivisions (up to simplicial map bijections).

The **star** of a vertex v in K is $st(v, K) = \{\sigma \in K \mid v \in \sigma\}$. The **link** of v in K is $link(v, K) = \{\sigma - \{v\} \mid v \in \sigma\}$. A **simpicial n-manifold** is a simplicial complex K such that $|K|$ is an n-manifold, and for each $v \in K$, $link(v, K)$ is a simplicial $n-1$-sphere combinatorially equivalent to $\partial\Delta^n$.