## RESEARCH BLOG 2/28/03

The Ricci flow is defined by the equation $\frac{\partial}{\partial t} g_t = -2Ric(g_t)$, where $(M, g_t)$ is a (smooth) 1-parameter family of Riemannian metrics with fixed manifold $M$. It's unclear exactly how Hamilton decided to study this equation, maybe because it's the simplest to compute with. He states in his paper [2] that he first considered the equation $\frac{\partial}{\partial t} g_t = \frac{2}{n} R g_t - 2Ric(g_t)$, which is the gradient flow of the total scalar curvature, and if it exists, preserves the volume of the manifold. This was suggested by Eells and Sampson *(note added 5/20/03: it appears that Eells and Sampson do not consider the total scalar curvature functional, and that Hamilton's work is only "inspired" by their work)*, but this implies a backwards heat equation for the scalar curvature $R$, which apparently will not even have solutions for a short time (if anyone could explain this to me, I would appreciate it). I suspect that if one imposed enough conditions on the Riemannian metric, then one might get backwards solutions. One possible idea is to start with a real-analytic manifold with real-analytic metric $(M, g_0)$, and try to find conditions on $g_0$ which would imply that the solutions $(M, g_t)$ are analytic functions on $M \times \mathbb{R}$. Solutions to parabolic equations usually become analytic immediately, but won't be an analytic function of the time variable. Here's a conjecture: if for any $m \in M$, $exp_m^*(g)$ is an entire symmetric bilinear form on $T_m M$ (that is, the Taylor series for the metric pulled back to the tangent space converges on all of $T_m M$), then we can find analytic solutions to Ricci flow or the gradient flow of total scalar curvature. Every differentiable manifold has an analytic metric, but I don't know how restrictive it is to assume that the metric is entire on the tangent space. If a flow on metrics were analytic in time, then we could analytically continue backwards in time, at least for a little bit.

I was interested in trying to modify the Ricci flow, since the Ricci flow doesn't behave well on negatively curved metrics, and it would be nice to have a flow which preserved negative curvature. In fact, hyperbolic metrics are in some sense unstable fixed points of Ricci flow. So I wanted to be able to flow it backwards, to make hyperbolic manifolds stable fixed points. Since the gradient flow of total scalar curvature implied backwards heat equation of scalar curvature, I thought that maybe one would want to flow in reverse, *i.e.* $\frac{\partial}{\partial t} g_t = -\frac{2}{n} R g_t + 2Ric(g_t)$. I asked Hamilton about this, and he says:

*The Flow by gradient of the integral of the scalar curvature is parabolic in the direction of conformal change, neutral in the direction of the diffeomorphisms, and backward in the directions perpendicular. So it doesn't work forward or back. Anderson has tried a min-max procedure. Rugang Ye tried a modified Ricci flow keeping scalar curvature constant. It looks something like Navier-Stokes-hard to analyse because it is hard to use the maximum principle on mixed elliptic-hyperbolic systems. (Sept. 1, 2002)*

In December, I started thinking about Ricci flow again. I had sent an application to American Institute of Mathematics to run a workshop on the hyperbolization conjecture, and they wrote me back in December, saying that they would like to have the workshop, but that maybe in the light of Perelman's work , we could have some emphasis on Ricci flow. So Chow and Schoen accepted to organize as well, but right now the dates of the workshop are not worked out, since we scheduled it for May, but this happened to conflict with an Oberwolfach conference on exactly the same topic, inviting many of the same people! So we'll have to have it later, hopefully in December.

At first, I tried to find another equation to flow by. Given a Riemannian manifold $M$ with metric $g \in S^2T^*M$, which is a symmetric form on $TM \otimes TM$, the curvature tensor $Rm_g$ lies in $S^2 \wedge^2 T^*M$ (see 3.5, [1]). We may also obtain a symmetric tensor on $S^2 \wedge^2 T^*M$ by taking $g \cdot g$, where $\cdot$ represents the Kulkarni-Nomizu product (see 3.122, [1]). Given $h, k \in S^2T^*M$,

$$h \cdot k(x,y,z,t) = h(x,z)k(y,t) + h(y,t)k(x,z) - h(x,t)k(y,z) - h(y,z)k(x,t).$$

This is the natural way to extend a metric on $TM$ to tensor bundles. Given a 1-parameter family $g_t$ of metrics on $M$, $\frac{\partial}{\partial t}(g_t \cdot g_t) \in S^2T^*M$. We say that $g_t$ is given by the curvature flow if $\frac{\partial}{\partial t}(g_t \cdot g_t) = -4Rm_{g_t}$. I wanted to analyze when such a curvature flow exists. In the case of a 3-dimensional manifold $M^3$, the curvature tensor is given by $Rm = (Ric - \frac{1}{4}Rg) \cdot g$, where $Ric$ is the Ricci curvature and $R$ is the scalar curvature (see 3.128, [1]). Since $\frac{\partial}{\partial t}(g \cdot g) = 2g \cdot \frac{\partial}{\partial t}g$, in 3-dimensions the curvature flow is equivalent to the flow $\frac{\partial}{\partial t}g = -2(Ric - \frac{1}{4}Rg)$. Hamilton's Ricci flow $\frac{\partial}{\partial t}g = -2Ric$ in 3-dimensions [2] is equivalent to the flow $\frac{\partial}{\partial t}(g \cdot g) = -4Rm_g - R(g \cdot g)$. The first difficulty is to establish short time existence. This seems to be a weakly parabolic equation, but Hamilton has to work some magic using the Nash-Moser implicit function theorem in order to prove short-time existence. It seems that the maximal $c$ for which the operator $cRg - 2Ric(g)$ is weakly elliptic is $\frac{1}{2}$, so when $c = \frac{2}{3}$, the equation is not weakly parabolic. De Turck's trick might work in this case, but I don't really understand his method, and the computations get intricate quickly. The evolution of curvature under this equation is not as nice either. So I've given up on analyzing this equation, but in the process I learned how to manipulate some curvature computations using Einstein summation convention. Hamilton said one time in his class that in differential geometry, one needs to have the most efficient notation in order to produce theorems quicker, otherwise other people will prove them first. Unfortunately, the notation can make it hard to get a start in the field. 3-manifold topology must not be as competitive, since we use such unwieldly terminology as "sutured manifold hierarchy" and "boundary incompressible essential surface".

The main motivation for studying Ricci flow then seems to be analytic: one can prove weak parabolicity, and in fact the equation is equivalent to a strictly (quasi-linear) parabolic equation, using DeTurck's gauge-fixing trick. Also, the curvature evolves by a non-linear heat equation, where the Laplacian is the Laplacian of the metric, so changes with time. Thus, one can use methods of parabolic PDE's to analyze the curvature, e.g. use the maximum principle. A maximum principle argument allows Hamilton to show that positive Ricci curvature is preserved by the Ricci flow in dimension 3. I should remark that Hamilton has completely revamped his original paper on manifolds with positive Ricci curvature, and the argument seems to be much simplified, although still quite non-trivial. Hamilton gives another motivation-by-analogy reason for studying the Ricci flow. In the paper "The formation of singularities in the Ricci flow" [3], Hamilton claims that "$\Delta$"$g_{ij} = g^{pq}\frac{\partial^2}{\partial x^p \partial x^q} g_{ij} = -2R_{ij}$. This is in local geodesic coordinates, where $ds^2 = g_{ij}dx^i dx^j$ and $R_{ij}$ is the $ij$ entry of the Ricci tensor, and presumably the equation is only supposed to hold at the origin. I think what he means by local geodesic coordinates is local normal coordinates, which are pull-back coordinates under the exponential map, in some neighborhood of the point where the exponential map is non-degenerate. Then "$\Delta$" is the laplacian in the euclidean metric on the tangent space (the usual laplacian of $g$ is always 0, since the covariant derivative $\nabla g = 0$, by definition, so $\mathrm{tr}\nabla^2 g = 0$ ). In this case, it seems this equation is incorrect. I tried computing this, and there are terms that give "$\Delta$"$g$, but there are other terms that don't seem to go away. I computed this explicitly for the standard metric on $S^2$, and I don't get the right answer (although I could easily have made a mistake, since the computation is quite involved - the metric in normal coordinates can be surprisingly complicated!). There is a more general reason why one would expect this equation to be invalid. Assume that we have chosen normal coordinates such that the Ricci tensor is diagonalized at the origin, so $R_{ij} = 0$ if $i \neq j$. If the equation held in two dimensions, for which $Ric = \frac{1}{2}Rg = Kg$ (K is the sectional curvature), then in $n$ dimensions, we would have

$$R_{ii} = \sum_{j\neq i} R_{ijij} = C \cdot \sum_{j \neq i} (\frac{\partial^2}{\partial_i^2} + \frac{\partial^2}{\partial_j^2}) g_{ii} \neq \sum_j \frac{\partial^2}{\partial_j^2} g_{ii} = \text{``}\Delta\text{''} g_{ii}.$$

Here, we have used the fact that normal coordinates restricted to a 2-dim. subspace are normal coordinates on that subspace. If these objections are well-founded, then $Ric$ appears to behave like an elliptic operator, but is not formulated as a Laplacian in this fashion.

## References

[1] S. Gallot, D. Hulin, and J. Lafontaine. *Riemannian geometry.* Springer-Verlag, Berlin, second edition, 1990.

[2] R. S. Hamilton. Three-manifolds with positive Ricci curvature. *J. Differential Geom.*, 17(2):255–306, 1982.

[3] R. S. Hamilton. The formation of singularities in the Ricci flow. In *Surveys in differential geometry, Vol. II (Cambridge, MA, 1993)*, pages 7–136. Internat. Press, Cambridge, MA, 1995.