RESEARCH BLOG 3/06/03

Elizabeth Klodginski asked whether the injectivity radius of closed 3-manifolds with non-positively curved (npc) cubings has a lower bound (n.b.: she has accepted a VRAP at UC Davis for next year). Her motivation was to find explicit examples of fibered manifolds which do not have an npc cubing, since one can find fibered hyperbolic 3-manifolds of arbitrarily small injectivity radius. So a lower bound on the injectivity radius of a cubed manifold would give examples of fibered manifolds that aren't cubed. I believe this question came out of studying a construction of Aitchison and Rubinstein of cubed 3-manifolds which are virtually fibered [2]. Liz says that she discovered a problem with their argument (which is pretty sketchy), but she has shown that their method works sometimes by working out an explicit example. Recently, as part of her thesis, she has also shown that surfaces in fibered hyperbolic 3-manifolds which are transverse to the pseudo-Anosov flow do not satisfy the 1-line condition (so in particular, they could *not* be the canonical π_1-injective surface associated to a cubing), answering a question of Alan Reid. She gave a talk at UIC on this work, which makes nice use of the dynamics of the pA map and work of Cooper, Long, and Reid on surfaces transverse to pA flows of fibered manifolds [3].

Work of Tao Li [4] shows that given M an orientable and irreducible 3-manifold whose boundary is an incompressible torus, and such that M does not contain any closed nonperipheral embedded incompressible surfaces, then only finitely many Dehn fillings of M can yield 3-manifolds with npc cubings. Given this statement, it seems natural to conjecture that there might be only finitely many cubed manifolds of bounded volume, or more generally, that there might be a lower bound on the injectivity radius of a cubed manifold. But by work of Aitchison, Lumsden, and Rubinstein [1], the complements of prime alternating links have npc cubings. If one takes a reduced alternating diagram of a prime alternating link, then the two checkerboard surfaces are π_1-injective. Taking the two checkerboard surfaces and a pushed off copy of the boundary torus, one obtains a π_1-injective surface satisfying the 4-plane, 1-line, triple point condition, with complementary regions balls. Thus, it induces a cell structure such that the dual cell structure is an npc cubing. But there are infinitely many alternating links of bounded volume. I believe one can use this fact to construct infinitely many closed npc cubed hyperbolic 3-manifolds of bounded volume, and therefore with injectivity radius approaching 0. To see this, one may take the Whitehead link and create an orbifold by adding an arc from one boundary component to itself, and making it into a π-orbifold axis, and putting a mirror on this boundary. This

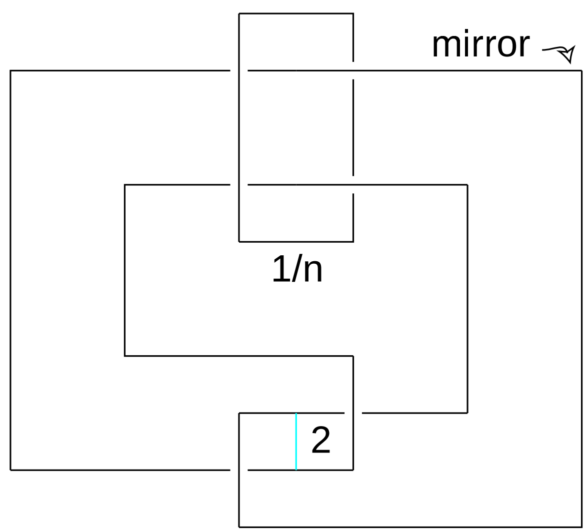

Figure 1. A family of cubed orbifolds

orbifold should be hyperbolic, with a cusp and a geodesic boundary component. Performing $1/n$ Dehn filling on the cusp gives the twist knots, which are also cubed. The orbifold axis is isotopic to a pair of edges in the cubings of these twist knots, so one gets a cubing of the associated orbifold of bounded volume for each twist knot. Taking a 4-fold orbifold cover which is a manifold gives infinitely many cubed manifolds of bounded volume (this cover comes by doubling along the boundary, then taking the double branched cover over the doubled π-orbifold axis). This does not violate Tao Li's theorem, since we are essentially performing Dehn filling on several boundary components of the orbifold cover.

References

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[4] T. Li. Boundary curves of surfaces with the 4-plane property. *Geom. Topol.*, 6:609–647 (electronic), 2002.