RESEARCH BLOG 4/1/03

Walter Neumann discussed the Casson invariant conjecture at the Spring Topology and Dynamical Systems Conference. This conjecture states that for a homology sphere Σ^3 which is the link of a complete intersection complex surface singularity, the Casson invariant $\lambda(\Sigma)$ is $\frac{1}{8}$ the signature of the Milnor fiber. I'm not going to describe this conjecture, but apparently Neumann and his collaborator Jonathan Wahl have made some progress on this (see the papers at Neumann's web page) by computing both the Casson invariant and the signature of the Milnor fiber for a special class of examples. Chris Herald gave a talk generalizing the Casson invariant, using a count of flat $SU(3)$ connections, and his attempt to generalize this to $SU(n)$. I don't really understand this stuff too well, but it would be nice to have a more topological way of defining the Casson and related invariants. Hyam Rubinstein and Weiping Li showed that the Casson invariant is a homotopy invariant.

One possible way one might be able to get the Casson invariant in a topological fashion is the following: given a homology sphere Σ^3, consider the maps $\rho : \pi_1(\Sigma) \to \mathrm{SU}(2)$. Then for each such ρ, one gets an element $h_\rho \in H_3(\mathrm{SU}(2), \mathbb{Z})$. To see this, consider a $K(\mathrm{SU}(2), 1)$ classifying space. This is a space with $\pi_n = 0$ for $n > 1$, and $\pi_1 \cong \mathrm{SU}(2)$, which is well-defined up to homotopy. Then for any map $\rho : \pi_1(\Sigma) \to \mathrm{SU}(2)$, we obtain a map $z_\rho : \Sigma \to \mathrm{SU}(2)$, which induces the map ρ on fundamental group. Then $[z_\rho] \in H_3(K(\mathrm{SU}(2), 1), \mathbb{Z}) = H_3(\mathrm{SU}(2), \mathbb{Z})$ is a canonically defined homology class, since z_ρ is well-defined up to homotopy. Notice that this construction works for any group G. What I conjecture is that there is a function on $\lambda : H_3(\mathrm{SU}(2), \mathbb{Z}) \to \mathbb{Z}$ which "counts" homology classes. For a homology sphere Σ with a discrete set of reps into SU(2), then the Casson invariant $\lambda(\Sigma)$ is the sum of $\lambda(h_\rho)$, over all representations $\rho : \Sigma \to \mathrm{SU}(2)$, so λ would be a sort of universal Casson invariant. If Σ is non-Haken, then the set of representations is finite, so this should make sense. I'm not sure how to deal with curves of representations, though (but of course I have no idea whether any of this works - it's probably way too simple-minded). If something like this were true, then the Casson invariant would be manifestly homotopy invariant, and one might have an idea how to generalize it to other Lie groups G, by considering $H_3(G,\mathbb{Z})$.

In the paper *Hilbert's 3rd problem and invariants of 3-manifolds*[3], Neumann discusses the relation between $H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})$ and Dehn invariants. Given a closed orientable 3-manifold M, as noted above, a representation $\rho:\pi_1(M)\to \mathrm{PSL}(2,\mathbb{C})$ gives an element $h_\rho\in H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})$. This is related to scissors congruence in $\mathbf{H}^3$ and in S^3. Complex conjugation induces an involution of $H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})$, with eigenspaces $H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})^{\pm}$ with eigenvalues ± 1. Then

$$H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})^{+}\cong H_3(\mathrm{PSL}(2,\mathbb{R}),\mathbb{Z}),$$

and

$$H_3(\mathrm{PSL}(2,\mathbb{C}),\mathbb{Z})^{-}\cong H_3(\mathrm{SU}(2),\mathbb{Z}).$$

Given an element $[z]\in H_3(\mathrm{PSL}(2,K),\mathbb{R})$ where K is a number field, there is the Borel regulator map. I think one way to see this map is to take a map of $K(\mathrm{PSL}(\tilde{2},\mathbb{C}),1)\to\mathbf{H}^3$ equivariant under the action of $\mathrm{PSL}(2,\mathbb{C})$. Then if we take a singular cycle representative of the homology class $[z]$, we may map the simplices to $\mathbf{H}^3$ and compute their (signed) volume. If $[z]\in H_3(\mathrm{PSL}(2,K),\mathbb{R})$, where K is a number field, then for each complex embedding $K\hookrightarrow\mathbb{C}$, one gets an associated volume. If K has σ embeddings into $\mathbb{C}$ (up to complex conjugation), then one gets a map $H_3(\mathrm{PSL}(2,K),\mathbb{R})\to\mathbb{R}^{\sigma}$, called the *Borel regulator map*, which computes the volume of the cycle for each embedding $K\hookrightarrow\mathbb{C}$. Borel showed that this gives an isomorphism. There is an associated norm on $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{R})$, where for each complex embedding $\overline{\mathbb{Q}}\hookrightarrow\mathbb{C}$, one has an associated volume, giving an embedding $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{R})\to\mathbb{R}^{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}$. Then one takes the l_∞ norm on this embedding. If one has a closed orientable hyperbolic 3-manifold, then the discrete faithful representation gives a homology class in $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{Z})$, and the norm of the homology class give the volume of the manifold, by the volume rigidity theorem (see [2] for proof of volume rigidity). In fact, Neumann and Yang have shown that there is also a well-defined homology class for a finite-volume hyperbolic 3-manifolds [4]. A bold conjecture is that the l_∞ norm on $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{Z})$ is minimized by the homology class represented by the Weeks manifold. This conjecture would imply that the Weeks manifold is the smallest volume hyperbolic 3-manifold. I pose this as a conjecture, only to point out how little is known about these homology groups. Borel has a formula for the covolume of the image lattice $H_3(\mathrm{PSL}(2,K),\mathbb{Z}) \to H_3(\mathrm{PSL}(2,K),\mathbb{R})$, up to a rational constant, where K is a number field. The Lichtenbaum conjectures state what the rational constant should be for certain number fields. If the field K were Galois over $\mathbb{Q}$, then the Galois group action on $H_3(\mathrm{PSL}(2,K),\mathbb{Z})$ would break up into rationally irreducible actions on the lattice $H_3(\mathrm{PSL}(2,K),\mathbb{Z})$, and there ought to be corresponding conjectures for the covolumes of these lattices. This would in turn give lower bounds on the length of vectors, since the orbits of a vector would span the irreducible subspace. Then one might be able to generalize techniques of Odlyzko [5] to estimate the covolumes of these lattices, and therefore get lower bounds on the l_∞ norms of homology classes, giving a strategy for approaching the conjecture. A similar strategy might work for analyzing Mahler measure of polynomials, which hopefully I'll discuss another time.

Given a hyperbolic 3-manifold, one may compute its hyperbolic volume to arbitrary accuracy. But given two hyperbolic 3-manifolds, how do we detect if they have the same volume? If the volumes are distinct, then the volume computations will differ at some stage. One can show that two hyperbolic 3-manifolds have the same volume, by showing that they are mutants, or have common finite sheeted covers of the same index which are isometric. Or one could try to show that they are scissors congruent, that is that one can cut one of the hyperbolic manifolds into a bunch of polyhedra, and reassemble them to form the other one. Given Neumann's theorem [3], this would imply that the two 3-manifolds represent the same homology class in $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{Z})^-$. A conjecture of Ramakrishan would imply the converse: if two hyperbolic 3-manifolds have the same volume, then they represent the same homology class in $H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{Z})^-$. Assuming this conjecture, I think there ought to be an algorithm to tell if two hyperbolic 3-manifolds have the same volume. First, we would find a ring R which is of the form $\mathcal{O}[1/n]$, where $\mathcal{O}$ is the ring of integers in a number field K, and $n \in \mathcal{O}$, such that both manifolds represent homology classes in $H_3(\mathrm{PSL}(2,R),\mathbb{Z})$. I believe that we should have an injection $H_3(\mathrm{PSL}(2,R),\mathbb{Z}) \to H_3(\mathrm{PSL}(2,\overline{\mathbb{Q}}),\mathbb{Z})$. If so, then we would only have to test the equality of homology classes in $H_3(\mathrm{PSL}(2,R),\mathbb{Z})$. Use Selberg's lemma to find a finite index torsion-free subgroup $G \leq \mathrm{PSL}(2,R)$. By taking all embeddings of $R \hookrightarrow \mathbb{R}$ or $\mathbb{C}$, and valuation trees for each prime $p|n$, we get actions of $\mathrm{PSL}(2,R)$ on symmetric spaces or trees. The product of these actions should be discrete and of cofinite volume. A torsion-free subgroup G should act freely, and the quotient should be a $K(G,1)$. Then we should be able to compute a cell-complex structure for this $K(G,1)$, and then compute the homology groups $H_3(G,\mathbb{Z})$. These should be finite index in $H_3(\mathrm{PSL}(2,R),\mathbb{Z})$, so we can take common multiples of our homology classes lying in $H_3(G,\mathbb{Z})$, and see whether they are the same when projected to $H_3(G,\mathbb{Z})^-$. Ramakrishnan's conjecture seems extraordinarily difficult, but assuming it, this outline seems like a plausible method to compute equality of volumes.

The reason I am interested in this question is that there ought to be an algorithm to find the smallest volume hyperbolic 3-manifold. Using the Margulis lemma, and the Jorgensen-Thurston decomposition of hyperbolic 3-manifolds, one ought to be able to algorithmically construct a finite list of hyperbolic 3-manifolds which are candidates for being minimal volume. But the difficulty is that if there are two minimal volume manifolds which are not isometric, then we need to be able to tell if they have the same volume. If there is only one minimal volume hyperbolic 3-manifold, as conjectured, then this is not a problem. For example, Cao and Meyerhoff [1] showed that the figure eight knot complement and its sibling are minimal volume orientable 1-cusped hyperbolic 3-manifolds. But it is easy to check that they have the same volume, since they both decompose into two regular ideal tetrahedra.

References

[1] C. Cao and G. R. Meyerhoff. The orientable cusped hyperbolic 3-manifolds of minimum volume. *Invent. Math.*, 146(3):451–478, 2001.

[2] N. M. Dunfield. Cyclic surgery, degrees of maps of character curves, and volume rigidity for hyperbolic manifolds. *Invent. Math.*, 136(3):623–657, 1999.

[3] W. D. Neumann. Hilbert's 3rd problem and invariants of 3-manifolds. In *The Epstein birthday schrift*, volume 1 of *Geom. Topol. Monogr.*, pages 383–411 (electronic). Geom. Topol. Publ., Coventry, 1998.

[4] W. D. Neumann and J. Yang. Invariants from triangulations of hyperbolic 3-manifolds. *Electron. Res. Announc. Amer. Math. Soc.*, 1(2):72–79 (electronic), 1995.

[5] A. M. Odlyzko. Bounds for discriminants and related estimates for class numbers, regulators and zeros of zeta functions: a survey of recent results. *Sém. Théor. Nombres Bordeaux (2)*, 2(1):119–141, 1990.