Last Wednesday, Howie Masur gave a talk on the curve complex of handlebodies. He and Yair Minsky have answered a question which I wondered about when I first saw their papers on the hyperbolicity of the curve complex (see [1]). Given a surface S (we'll assume it's closed, for simplicity), the 1-skeleton of the curve complex $C(S)$ has vertices consisting of isotopy classes of simple closed curves on the surface, and edges between curves which are disjoint (and distinct). If $S = \partial H$ is the boundary of a handlebody, then there is a subset $C(H) \subset C(S)$ consisting of the subcomplex spanned by curves in S which bound disks in H. What Masur and Minsky proved is that $C(H)$ is quasiconvex in $C(S)$, that is every geodesic in $C(S)$ connecting vertices of $C(H)$ is within a bounded distance $K = K(S)$ of $C(H)$ [2]. For each pair of vertices $c_1, c_2 \in C(H)$, they find a path in $C(H)$ connecting c_1 and c_2 which is a K' quasi-convex subset of $C(S)$, from which the theorem follows. Since c_1 and c_2 bound disks in H, the usual outermost disk argument shows that c_1 will have a wave with respect to c_2. One does surgery along this wave to get a curve c' disjoint from c_2 which has fewer intersections with c_1 and still bounds a disk in H, so is a vertex of $C(H)$. Continuing carefully in this fashion, they produce a quasi-convex path in $C(H)$. I was originally interested in this question because I thought it might enable one to detect if a Heegaard splitting of a 3-manifold is reducible. If $S = \partial H_1 = \partial H_2$ is a Heegaard splitting for a 3-manifold, then one would like to know if there is a curve in S which bounds disks in both H_1 and H_2. This can be restated as $C(H_1) \cap C(H_2) \neq \emptyset$. Since $C(S)$ is δ-hyperbolic, and $C(H_i)$ is a quasiconvex subset, one might hope that starting with arbitrary vertices c_i of $C(H_i)$, there is some process which would eventually find a pair of curves c_i' which realize the minimal distance $d(C(H_1), C(H_2))$. Of course, things aren't that simple since the curve complex is not locally finite. Saul Schleimer and Howie Masur have been consider whether one can even compute $d(C(H_1), C(H_2))$ up to some constant factor, but haven't succeeded yet. Apparently, a student of Minsky has shown that if $d(C(H_1), C(H_2))$ is large enough (something like $2K+2\delta$), then the manifold has finite mapping class group, using a result of Jaco and Rubinstein that an atoroidal manifold has only finitely many Heegaard splittings of bounded genus (if $d(C(H_1), C(H_2)) > 2$, then the manifold is atoroidal).

Yesterday, we had two talks at UIC. Michael Handel discussed the commensurator of the outer automorphisms of a free group. He and Benson Farb think that they can show that if $\Lambda \leq Out(F_n)$ is finite index, then any other embedding $\Lambda \hookrightarrow Out(F_n)$ extends to an inner automorphism of $Out(F_n)$. This would imply that $Comm(Out(F_n)) = Out(F_n)$.

Damien Gaboriau gave a colloquium on orbit equivalence for group actions. Given two groups Γ_i acting on a probability measure space $\mathcal{M}$ with no atoms (the "only" such measure space is $[0,1]^{\mathbb{N}}$, with product of counting measure), the actions are orbit equivalent if there is a measurable map $\mathcal{M} \to \mathcal{M}$ taking the orbits of Γ_1 to the orbits of Γ_2. Free ergodic actions by amenable groups are all orbit equivalent. But Gaboriau shows that if Γ_1 and Γ_2 have orbit equivalent free ergodic actions, then $\beta_i^{(2)}(\Gamma_1) = \beta_i^{(2)}(\Gamma_2)$. These are the ℓ_2 betti numbers, which I won't define, but just mention that these are real numbers which measure the von Neumann dimension of some sort of homology associated with Γ_i. But if $\chi(\Gamma_i)$ is well-defined, then a corollary is that $\chi(\Gamma_1) = \chi(\Gamma_2)$, since $\chi^{(2)}(\Gamma_i) = \chi(\Gamma_i)$, when well-defined. For fundamental groups of 3-manifolds, I suppose this result is only interesting when the manifold has boundary of genus > 1, in which case the euler characteristic is non-zero, and could distinguish orbit-equivalent actions. There seems to be quite a lot of work studying orbit equivalence lately, as it is a natural extension of various rigidity results.

At the U. of Arkansas Spring Lecture series, Alan Reid gave an interesting talk on the arithmetic rational homology spheres (slides available). An arithmetic hyperbolic manifold M is a finite volume hyperbolic manifold for which the discrete faithful representation $\pi_1(M) = \Gamma \leq \mathrm{PSL}(2,\mathbb{C})$ may be conjugated into $\Gamma \leq \mathrm{PSL}(2,F)$, where F is a number field, such that the traces lie in $\mathcal{O}(F)$, the ring of integers of F, and for any other embedding of $\sigma : F \hookrightarrow \mathbb{C}$, $\sigma(\Gamma) \leq \mathrm{SU}(2)$. As the name suggests, arithmetic manifolds have special constructions coming from number theory and algebraic groups. A conjecture is that any hyperbolic 3-manifold M has a finite sheeted cover $\tilde{M}$ for which $\beta_1(\tilde{M}) > 0$. In the case of finite volume cusped manifolds, one can ask for cuspidal cohomology, so if M is compact with boudary, and $int(M)$ is hyperbolic, one wants a finite sheeted cover $\tilde{M}$ such that $\beta_1(\tilde{M}, \partial\tilde{M}) > 0$, that is there is non-trivial homology which doesn't come from the boundary. One can also consider this in the fuchsian case, where M is a 2-dimensional finite volume hyperbolic orbifold, in which case cuspidal cohomology corresponds to having genus > 0. Long, Reid, and Sarnak show that there are only finitely many arithmetic fuchsian orbifolds of genus 0. This comes from remarkable properties of the eigenvalues of the Laplacian. Reid also observes that for various congruence covers of arithmetic manifolds (covers for which the matrix entries of Γ satisfy congruence conditions in $\mathcal{O}(F)$), one may use the Jacquet-Langlands correspondence to show that cuspidal cohomology of cusped arithemetic congruence manifolds may be transferred to positive betti numbers of closed arithmetic manifolds. Reid also asks whether one may have rational homology spheres of arbitrarily large injectivity radius. He experimentally finds some with injectivity radius > 1.25. More generally, it would be interesting to know if there are non-Haken 3-manifolds of arbitrarily large injectivity radius.

References

[1] H. A. Masur and Y. N. Minsky. Geometry of the complex of curves. I. Hyperbolicity. *Invent. Math.*, 138(1):103–149, 1999. math.GT/9804098.

[2] H. A. Masur and Y. N. Minsky. Quasiconvexity in the curve complex. preprint, 2003.