

RESEARCH BLOG 5/6/03

I haven't heard much about Perelman's lectures at Stonybrook, only that his work he's written so far seems to be holding up well, and he is not discussing the collapsing part of his argument which is unwritten, but is the type of thing on which he is an expert.

At the U. of Arkansas Spring Lecture Series, Marty Scharlemann spoke about his solution to the Goda-Teragaito conjecture, which gives a complete classification of tunnel number one, genus one knots in S^3 . That is, knots which have a genus 2 Heegaard decomposition, such that the knot bounds a punctured torus. There are satellite examples, which were classified by Eudave-Munoz and by Morimoto-Sakuma, as well as two-bridge knots which are plumbings of unknotted annuli. His argument makes much use of thin position for graphs in S^3 and other techniques developed by Scharlemann-Thompson, and thus seems rather special to knots. But it would be interesting to obtain a better understanding of tunnel number one knots and links in general. Marc Lackenby has classified tunnel number one alternating knots, using almost normal surfaces. I reckon his arguments could be generalize to classify all tunnel number one alternating links. One step in Scharlemann's argument is to make the tunnel disjoint from a Seifert surface for the knot. I think this may be done in some generality, with restrictions on the Seifert surface. Any Heegaard genus 2 manifold has a hyper-elliptic involution (an easy exercise checks that the hyper-elliptic involution of a genus 2 surface preserves any embedded curve (reversing orientation on the non-separating curves, and rotating the separating curves by π), which is why it is in the center of the mapping class group, since it commutes with Dehn twists, and is why it extends over any compression body bounding the surface, so extends to the whole manifold). Thus, a genus two manifold (which can have torus boundary components) has a hyper-elliptic involution, which reverses the orientation of every curve on the boundary tori (since these correspond to non-separating curves on the Heegaard surface). Thus, the link is strongly invertible.

So any homology class is sent to its negative under the strong involution. Then it should follow that there is a minimal Thurston norm surface representing the homology class which is taken disjoint from itself by the involution (I believe this was shown by Tollefson). If one has a tunnel number one link, there there is an unknotting tunnel, which is the core of the 1-handle for a compression body. I believe I can show that this tunnel may be isotoped off such an involution invariant pair of minimal Thurston norm surfaces (which are also required to be pairwise incompressible: there is no non-peripheral curve in the surface cobounding an annulus with a curve in the boundary of the manifold). Moreover, the region in between these surfaces which contains the tunnel has a very specific structure, with solid torus guts and the core of the tunnel going through the core of the guts. Thus, if the surface is fibered, for example, then the tunnel must be parallel onto the Seifert surface. Moreover, a similar argument should work for pairwise incompressible invariant surfaces coming from actions on trees associated to the ideal points of the character variety. In general, it is unknown whether the character variety detects all boundary slopes of surfaces. I think one might be able to show that the character variety detects Conway spheres (embedded 4-punctured spheres) in hyperbolic cusped manifolds, using the proof of the orbifold theorem. When one deforms from cone-angle infinity to π along the sphere boundary slope, the 4-punctured sphere transforms to a euclidean orbifold, which in the limit should correspond to the ideal point of the character variety.

If one has a manifold M with a smooth taut foliation \mathcal{F} , then one can put a Riemannian metric g on M such that the leaves of \mathcal{F} are minimal surfaces, by a result of Sullivan. Suppose one had a map $f : S \rightarrow M$ from a surface S which is minimal in the metric g . Then either the induced foliation $f^*(\mathcal{F})$ has only isolated singularities, which may be degenerate, but are not center tangencies, and thus contribute negative index to the euler characteristic of $f^*(\mathcal{F})$, or $f(S) \subset F \in \mathcal{F}$, a map to a leaf of the foliation [3]. Thus, if \mathcal{F} is not a foliation with S^2 leaves, then $\chi(S) \leq 0$, and we see that (M, g) may not contain an immersed minimal 2-sphere. If $\chi(S) = 0$, then we see that either S is a cover of a compact leaf of \mathcal{F} , in which case M is toroidal, or f is transverse to \mathcal{F} . In this

case, either the intersection of $f(S)$ with each leaf is homotopically trivial, in which case I'm not sure if f can be stable, or f is a π_1 -injective map, and M is toroidal. If $\pi_1(M)$ is not word-hyperbolic, then unpublished work of Bruce Kleiner shows that there is a map of a stable minimal quasi-euclidean measured lamination $q : E \rightarrow M$. This generalizes work of Gabai in the normal surface category [2], and is based on an idea of Gromov (worked out for 2-complexes by Mosher and Oertel) which works for more general spaces, if one allows branching in the map. I believe a similar sort of argument should show that q is transverse to the leaves of \mathcal{F} . One would hope that stability of q would imply that the intersections with each leaf are non-homotopically trivial, and then try to do some sort of leafwise cut and paste to obtain an essential torus transverse to \mathcal{F} . This might give an alternative to an argument of Calegari showing weak hyperbolization for manifolds with taut foliations. One might be able to work in the topological category of taut foliations, by working with normal surfaces instead.

Last Wednesday, Saul Schleimer discussed examples relating Heegaard theory and the action on various complexes of the mapping class group. He shows that the actions of the mapping class group on the Cayley graph, the pants complex, and the curve complex are all distinct, even up to quasi-isometric equivalence. For the Cayley graph and pants complex, this is straightforward: take Dehn twist generators, then compose a pseudo-Anosov with powers of a Dehn twist generator. For large enough powers, we get pseudo-Anosov maps, where the pants complex distance is bounded, but the Cayley graph distances approaches ∞ .

To compare the actions on pants complex and curve complex, he constructs fibered manifold of Heegaard genus ≤ 3 with volume approaching ∞ . A result of Jeff Brock then shows that the pants distance approaches infinity, while a result of Bachman and Schleimer [1] shows that the curve complex distance remains bounded (although for the particular examples, I believe it is not hard to bound the distance directly). Here's a description of the construction: take a regular octagon centered at the origin in \mathbb{R}^2 such that every other side is perpendicular to the x and y coordinate axes, and take it $\times [-1, 1] \subset \mathbb{R}^3$. Then

glue opposite vertical faces of the resulting polyhedron, to get a genus 2 surface $\times [-1, 1]$. Glue the top to bottom with a $\pi/4$ rotation. This gives a Seifert fiber space, which fibers over S^1 with fiber the genus 2 surface. Any line of the form $(x, 0, c)$ or $(0, y, c)$ forms a closed curve in the resulting manifold. A regular neighborhood of the union of the x , y and z axes mapped to the manifold yield a Heegaard splitting of genus 3. Then one takes a regular neighborhood of the x and y axes in \mathbb{R}^2 to get a punctured torus subsurface. Modifying the gluing monodromy by a homeomorphism of this subsurface gives a new genus 2 fiber bundle which still clearly has Heegaard genus 3. If we have a hyperbolic fiber bundle, then the Heegaard splitting must be either strongly irreducible or a stabilization of a genus 2 splitting, since a weak reduction would yield an essential torus or sphere. Thus, the curve complex distance is bounded, by Bachman and Schleimer's result [1]. But a result of Minsky shows that the volume approaches infinity when one modifies the monodromy by powers of a pseudo-Anosov mapping class on the punctured torus subsurface, and therefore the pants complex distance of the mapping class goes to ∞ . One can also perform products of large powers of Dehn twists about the x and y axes to get fiber bundles of Heegaard genus 3 and volume approaching ∞ . To see this, drill out alternating curves parallel to the x and y axes. This gives a hyperbolic manifold with volume $\rightarrow \infty$ as the number of curves drilled $\rightarrow \infty$. Then one performs very high Dehn twists about these curves, to get fiber bundles of arbitrarily large volume, by Thurston's hyperbolic Dehn surgery theorem. In fact, one can choose these surgeries so that all the core curves have distinct lengths, all shorter than any other geodesic in the manifold. Then any involution of the manifold must preserve these curves. So if the manifold had Heegaard genus 2, then the hyper-elliptic involution would preserve these curves, and thus comes from an involution on the link complement. But the only such involution is the hyper-elliptic involution of the fiber bundle, which has quotient $S^2 \times S^1$. But the quotient of a hyper-elliptic involution of a Heegaard genus 2 manifold must be S^3 , so the manifolds could not be Heegaard genus 2 (in fact, the fundamental group must be 3 generator).

REFERENCES

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