Dan Knopf notified me that he has a survey paper (on which his talk at UIC was based) on models of singularities in the Ricci flow (see his web page).

I've been thinking about the conjectures made in research blog 3/4/03, specifically conjecture 2 about backwards propagation of minimal surfaces under Ricci flow, and conjecture 4 that in a bumpy metric, there are only finitely many disjoint embedded minimal 2-spheres (actually, I would rather know this for real-analytic Riemannian metrics, since under Ricci flow, the metric becomes instantaneously analytic). I'll review Hamilton's formula for the variation of area of minimal surfaces under Ricci flow, then I'll explain my strategy for attempting to prove the above two conjectures.

The following is an expanded version of pp. 40-41 of Hamilton's paper [3]. Consider a co-oriented surface Σ^2 in a Riemannian 3-manifold M^3 with metrics (M, g_t) satisfying the Ricci flow $\frac{\partial}{\partial t} g_t = -2Ric(g_t)$. At each point $x \in \Sigma$, choose an orthonormal frame for g_0 $\{e_1, e_2, N\}$, where $e_i \in T_x\Sigma$ and N gives the coorientation (we will suppress the dependence on x and g_t). Let $A_t = \int_\Sigma da_t = \int_\Sigma \sqrt{g_t(e_1,e_1)g_t(e_2,e_2) - g_t(e_1,e_2)^2} da$ denote the area of Σ with respect to the metric g_t. Then

$$\frac{\partial A}{\partial t}|_{t=0} = \int_\Sigma \frac{1}{2}(det(g))^{-\frac{1}{2}} \frac{\partial}{\partial t}(g_{11}g_{22} - g_{12}^2) da$$

$$= \int_\Sigma \frac{1}{2}(-2Ric_{11}g_{22} - 2Ric_{22}g_{11} + 4Ric_{12}g_{12} da$$

$$= \int_\Sigma -(Ric_{11} + Ric_{22}) da$$

$$= \int_\Sigma -2Rm(e_1,e_2,e_1,e_2) - Rm(e_1,N,e_1,N) - Rm(e_2,N,e_2,N) da$$

$$= \int_\Sigma -Ric(N,N) - 2Rm(e_1,e_2,e_1,e_2) da$$

(for a 2-tensor T, we abbreviate $T(e_i,e_j) = T_{ij}$).

Now, suppose that we take a 1-parameter family of surfaces $\Sigma_t \subset M$, moving with velocity V. Then the area changes at the rate

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} - \int_\Sigma g(trII, V)da$$

where II is the second fundamental form.

To see this, consider moving frames $\{e_1^t, e_2^t, N^t\}$. Since $g_t(e_i^t, e_j^t)$ is linear in g_t and in each e_k^t, we see by the product rule that the variation breaks up linearly into the variation due to the change in the metric and the variation due to the change in Σ_t. The second term gives the well-known formula for this variation (see *e.g.* 5.20, [2]).

So if we choose the family Σ_t so that it is a minimal surface in (M, g_t), then the second term disappears, and we obtain $\frac{dA}{dt} = \frac{\partial A}{\partial t}$.

Assume that Σ has an orientable normal bundle, and consider at a fixed time a one-parameter family of surfaces with parameter r starting at the given surface Σ at $r = 0$ and moving in the normal direction N with velocity 1. If Σ is minimal, then $\frac{\partial A}{\partial r} = 0$, and the second variation is given by the standard formula

$$\frac{\partial^2 A}{\partial r^2} = \int_\Sigma \{2\det B - Ric(N, N)\}da$$

where B is the real-valued second fundamental form (*i.e.* $II = BN$). The Gauss curvature K of Σ in the induced metric is given by

$$K = \det B + Rm(e_1, e_2, e_1, e_2); \int_\Sigma K da = 2\pi\chi(\Sigma),$$

the Gauss-Bonnet theorem. This gives the formula

$$\frac{\partial A}{\partial t} = \int_\Sigma -Ric(N, N) - 2Rm(e_1, e_2, e_1, e_2)da$$

$$= \int_\Sigma -Ric(N, N) + 2\det B - 2K da = \frac{\partial^2 A}{\partial r^2} - 4\pi\chi(\Sigma)$$

which is a "heat equation"! If Σ is a stable minimal surface (that is, $\frac{\partial A}{\partial t} > 0$), and $\chi(\Sigma) \leq 0$, then A is increasing with time. For example, this should hold true for minimal area representatives of incompressible surfaces in Haken manifolds, by a weak maximum principle (the minimal area surfaces might not vary continuously, but the minimal area

$\hat{A}$ should be Lipschitz and satisfy an inequality $\frac{\partial \hat{A}}{\partial t} \geq \frac{\partial^2 \hat{A}}{\partial r^2} - 4\pi\chi(\Sigma)$, suitably interpreted).

We would like to use this formula to show that minimal surfaces propagate backwards under the Ricci flow. That is, if Σ_T is a minimal surface in (M, g_T), then there should exist a continuous family of minimal surfaces Σ_t in (M, g_t), $t \leq T$. As I've mentioned before, this holds in the special case of warped product metrics for minimal surfaces which are warp factors. This is a rather special type of minimal surface, since it is totally geodesic. Yet I hope that the method for analyzing these might generalize in a certain sense.

Given a surface with metric of constant curvature $(\Sigma, d\sigma^2)$, the warp product is given by $(S^1 \times \Sigma, ds^2 \oplus \psi(s)^2 d\sigma^2)$, where s is the coordinate in the S^1 direction. The function $\psi(s)$ gives the size of the surface $\{s\} \times \Sigma$, and in these coordinates, the vector field $\frac{\partial}{\partial s}$ is the unit normal to the foliation by $\{s\} \times \Sigma$. If $\tilde{\Sigma}$ is the universal cover of Σ, then $\tilde{\Sigma}$ is homogeneous (either S^2, E^2 or $\mathbf{H}^2$), in fact completely isotropic, so $S^1 \times \tilde{\Sigma}$ is invariant under $Isom(\tilde{\Sigma})$. By the equivariance of Ricci flow under isometries, a solution to the Ricci flow $(S^1 \times \tilde{\Sigma}, g_t)$ beginning with a warp product metric will remain invariant under $Isom(\tilde{\Sigma})$, and therefore remains a warped product (although the vector field $\partial_s = \frac{\partial}{\partial s}$ does not remain unit length in g_t). The surface $\{s_0\} \times \Sigma$ will be minimal if and only if $\psi_s(s_0) = 0$, and in fact, by the homogeneity, will be totally geodesic. The profile function under Ricci flow satisfies an evolution equation

$$\psi_t = \psi_{ss} + \psi_s^2/\psi - K/\psi,$$

where $K = 1, 0, -1$ is the curvature of Σ. Since ∂_s does not remain invariant, we also have the equation

$$[\partial_t, \partial_s] = -2\psi_{ss}/\psi \partial_s.$$

The function $v = \psi_s$ also satisfies a parabolic equation, which is easily derivable from these two equations:

$$v_t = v_{ss} + K(1 - v^2)v/\psi^2.$$

By a result of Bando [1], the metric g_t is (real-)analytic for $t > 0$, and thus normal coordinates are anaytic. So the variable s is analytic, and one sees that $\psi(s)$ is an analytic function of s, so that $\psi_s(s)$ is also analytic, and thus its zeroes are discrete. Angenent and Knopf use this to show that the number of critical points of ψ decreases with time, by a Sturmian theorem due to Angenent. In fact, they show that if at a critical point at time t_0 $\psi_s(s_0) = \psi_{ss}(s_0) = 0$, then the number of critical points decreases when one passes through time t_0. Another way to describe this result is that critical points of ψ propagate backwards. I described a heuristic for this in research blog 3/4/03, since ψ^2 is essentially the area, its minimum satisfies a parabolic-type of equation which prevents a min-max birth of critical points. I discovered essentially the same result independently, but it is interesting that it holds for very general parabolic equations. This is what motivated me to conjecture that minimal surfaces propagate backwards under Ricci flow, and the nature of the proof in the warp-product case suggests an approach in the general case.

The surfaces $\{s\} \times \Sigma$ will be CMC surfaces, that is constant mean curvature (again, by the homogenity of $\{s\} \times \Sigma$). This suggests that one might try to consider a neighborhood of a minimal surface in a 3-manifold which has a foliation by CMC surfaces. I tried a literature search, but was unable to find any results of this type. But Hamilton uses a result of this type in his paper "Non-singular solutions of the Ricci flow on 3-manifolds" [4] without any proof or citation (and has so far not responded to an inquiry from me about this question, although I have a preprint version of the paper, and I have not checked to see if the published version contains a justification). One heuristic for believing such a foliation should exist is to consider a relative isoperimetric question. Given a co-oriented CMC surface $\Sigma \subset M$ with metric g, for a number $\epsilon \in \mathbb{R}$, consider

$$d(\epsilon) = \inf\{area(\Sigma') \subset M \mid [\Sigma'] = [\Sigma] \in H_2(M), M' \subset M,$$
$$\partial M' = \Sigma \cup -\Sigma', Vol(M') = \epsilon\}.$$

Then $d(\epsilon)$ the infimal area of surfaces Σ' which cobound a region M' with Σ of signed volume ϵ, which ought to be realized by a soap film. I conjecture that for $|\epsilon|$ small enough, $d(\epsilon)$ is realized by a CMC surface Σ_ϵ lying in a tubular neighborhood $\mathcal{N}_\delta(\Sigma)$, and such that for $\epsilon \neq \epsilon'$, Σ_ϵ

and $\Sigma_{\epsilon'}$ are disjoint, forming a foliation of a collar of $\Sigma_0 = \Sigma$. I also conjecture that this CMC foliation is unique. It might be possible to prove this conjecture using methods of geometric measure theory. If the metric g on M is real-analytic, then I conjecture that Σ_ϵ depends analytically on ϵ.

If this conjecture holds, then if g is analytic, then $A(\epsilon) = Area(\Sigma_\epsilon)$ should depend analytically on ϵ, and therefore one would expect only finitely many zeroes of $A'(\epsilon)$ and finitely many critical points of $A(\epsilon)$, or else $A(\epsilon)$ is constant, and one has a foliated neighborhood by minimal surfaces, in which case M should be a fiber bundle over S^1, by analytic continuation and the fact that limits of minimal surfaces are minimal surfaces. Thus, for a small enough CMC foliated collar, $\Sigma_0 = \Sigma$ would be the only minimal surface. Suppose (M, g) had infinitely many embedded disjoint minimal 2-spheres. By an argument of Joel Hass, there will exist a minimal 2-sphere F which is a limit of disjoint minimal 2-spheres F_i. Taking a CMC foliated collar $\Sigma_\epsilon, |\epsilon| < \delta$, $F = \Sigma_0$, such that for $\epsilon \neq 0$, the mean curvature of $\Sigma_\epsilon \neq 0$, we see that for i large enough, $F_i \subset \cup_{0<\epsilon<\delta}\Sigma_\epsilon$ (possibly by reparameterizing Σ_ϵ so that F_i approaches Σ_0 from the positive side). Then $F_i \subset \cup_{\epsilon\in[a,b]}\Sigma_\epsilon$, such that F_i is tangent to Σ_a and Σ_b, $0 < a < b < \delta$. But if the mean curvatures of Σ_ϵ are pointing towards F, we get a contradiction to F_i being tangent to Σ_b, since tangent CMC surfaces must have an inequality of mean curvatures, and similarly in the other case.

I would hope that if one runs Ricci flow to get metrics (M, g_t), $-\tau < t < \tau$, then given Σ^0_0 a minimal surface in (M, g_0), and a CMC foliated collar neighborhood Σ^0_ϵ, then one can find nearby foliated collar neighborhoods Σ^t_ϵ and a 1-parameter family of diffeos. ϕ_t such that $\phi_t(\Sigma^0_\epsilon) = \Sigma^t_{\epsilon'}$, and that the area $A^t_\epsilon = Area(\Sigma^t_\epsilon)$ might satisfy some sort of evolution equation, similar to that in the warp product case. Then one might be able to show local backward propagation of minimal surfaces in the CMC foliation, and use this to prove backwards propagation of minimal surfaces.

CMC collarings might also be useful for understanding minimal 2-spheres under surgery. If one has a neck pinch type singularity, then the metric should approach $ds^2 \oplus \epsilon^2 d\sigma^2$ on $[-x, x] \times S^2$, where $d\sigma^2$ is the standard metric on S^2, and $\epsilon \ll x$. Then one could hope to show that we may choose this parametrization so that $\{y\} \times S^2$ is CMC, and $\{0\} \times S^2$ is the only minimal surface in this collaring. Then one could try to prove a gluing theorem, that one may cut out $[-\epsilon, \epsilon] \times S^2$ and glue in two B^3's foliated by CMC spheres with a center singularity at a point, such that the mean curvatures are non-zero. This would imply that any new minimal 2-sphere would have to intersect the outside of the collared neighborhood, otherwise one would again get a contradiction to the maximum principle for CMC surfaces. If a new minimal 2-sphere intersected the region where the surgery is performed, then it would have to cross all the way across the neck, and one could hope to show that it is unstable.

References

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