MOSHER'S ARATIONALITY CRITERION

In Lee Mosher's talk at U. of Chicago on 2/27/03, he gave an example demonstrating how to tell if a lamination is *arational*. One way to define arational is that the lamination meets every simple closed curve essentially. On the torus, these correspond to measured laminations of irrational slope, so are a generalization of irrational numbers. A lamination may be specified by a sequence of splittings of train tracks, where each train track is some coarse view of the lamination, where one can resolve distinct leaves only up to a certain scale, after which leaves merge together at branches, called *cusps*. Mosher discussed a particular kind of sequence of splittings, where one has a distinguished cusp, marked by $*$. One splits the train track sequentially at the cusp $*$, and each splitting is specified by an L or R, depending on whether the branch splits to the left or right (see figure 1). When the two cusps agree, there is only one way to split, so we do not need to record it. If the train track fully carries a lamination (meaning that there is a smooth homotopy of the lamination into the train track), then this sequence of splittings is uniquely determined by the lamination. Thus, a sequence $\mathsf{LRLLRRRL}$... defines a unique sequence of splittings of the train track along the distinguished cusp.

Mosher starts with a particular train track on the 6 punctured sphere (which is drawn as 5-punctured $\mathbb{R}^2$). There are 5 monogon complementary regions, and one triangular region, each containing a single puncture (see the top train track in figure 2). L and R splittings give new train tracks. But when we perform LL splitting, we get a train track combinatorially equivalent to the starting track by a homeomorphism where we rotate three mongon regions counterclockwise 120^o about the tricusp region. Similarly, RL splitting produces a train track combinatorially equivalent to the R splitting, by a homeomorphism rotating the rightmost monogons 180^o counterclockwise about the tricusp. We get a finite-state automaton with three states corresponding to the 3

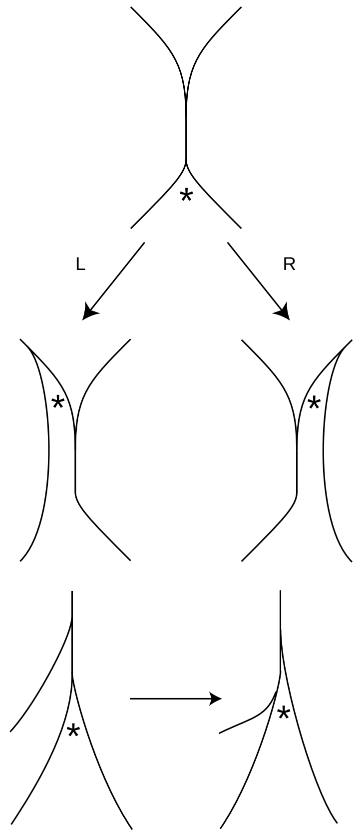

FIGURE 1. How to denote the splitting sequence

combinatorial types of train tracks, see figure 3, which we will label A, B, C.

There are arcs, containing an embedded interval of the branched surface and two cusps, with endpoints on punctures. In the train track A, we have drawn arcs A1, A2 connecting the upper monogon punctures to the trigon puncture (see figure 3). There are similar arcs A3, A4 connecting the lower monogon punctures to the trigon puncture. In train track B, there are 3 arcs B1, B2, B3 connecting monogon with trigon punctures, and one B4 connecting two monogon punctures. There are similar arcs in the C train track. Mosher's theorem states that a sequence of train track splittings gives an arational lamination if and only if every such arc does not survive under the sequence of splittings. One way to say this, is that any tail end of the path in the finite state

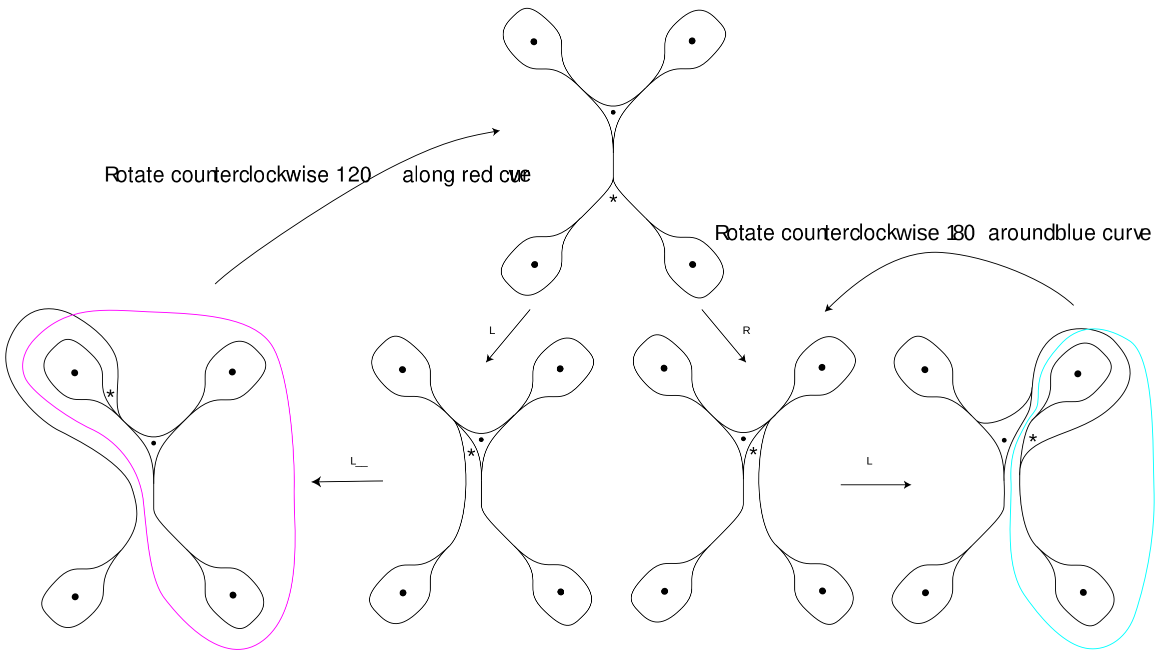

FIGURE 2. Various splittings

automaton track does not lift to a path in the finite state automaton arc which maps cusp arcs to cusp arcs under splittings (see figure 4).

For example, any sequence $\mathsf{L}^6\mathsf{R}^6\mathsf{R}^6\mathsf{L}^6\mathsf{L}^6$..., where L and R always occur with multiplicity 6 will not lead to an arational lamination, since the arc automaton has L and R cycles of length 6 starting at A1. The basic reason that these sequences are forbidden is that if we take an arc that survives all splittings, then the boundary of a little regular neighborhood of the arc gives a loop which is in the complement of the lamination. It is a theorem of Mosher that the converse is true - if all cusp arcs get killed, then the lamination is arational. Mosher also explained how to detect if a periodic lamination is arational. If the LR expansion is periodic, then it corresponds to a lamination fixed by a homeomorphism taking the initial train track to one at the period. The lamination is arational if and only if this homeomorphism is isotopic to a *pseudo-Anosov* homeomorphism. Mosher observes that one only need look at four periods of the splitting sequence, and see whether these lift from track to arc (since after four periods, one must have encountered the same node in arc, and then one can continue forever by repetition).

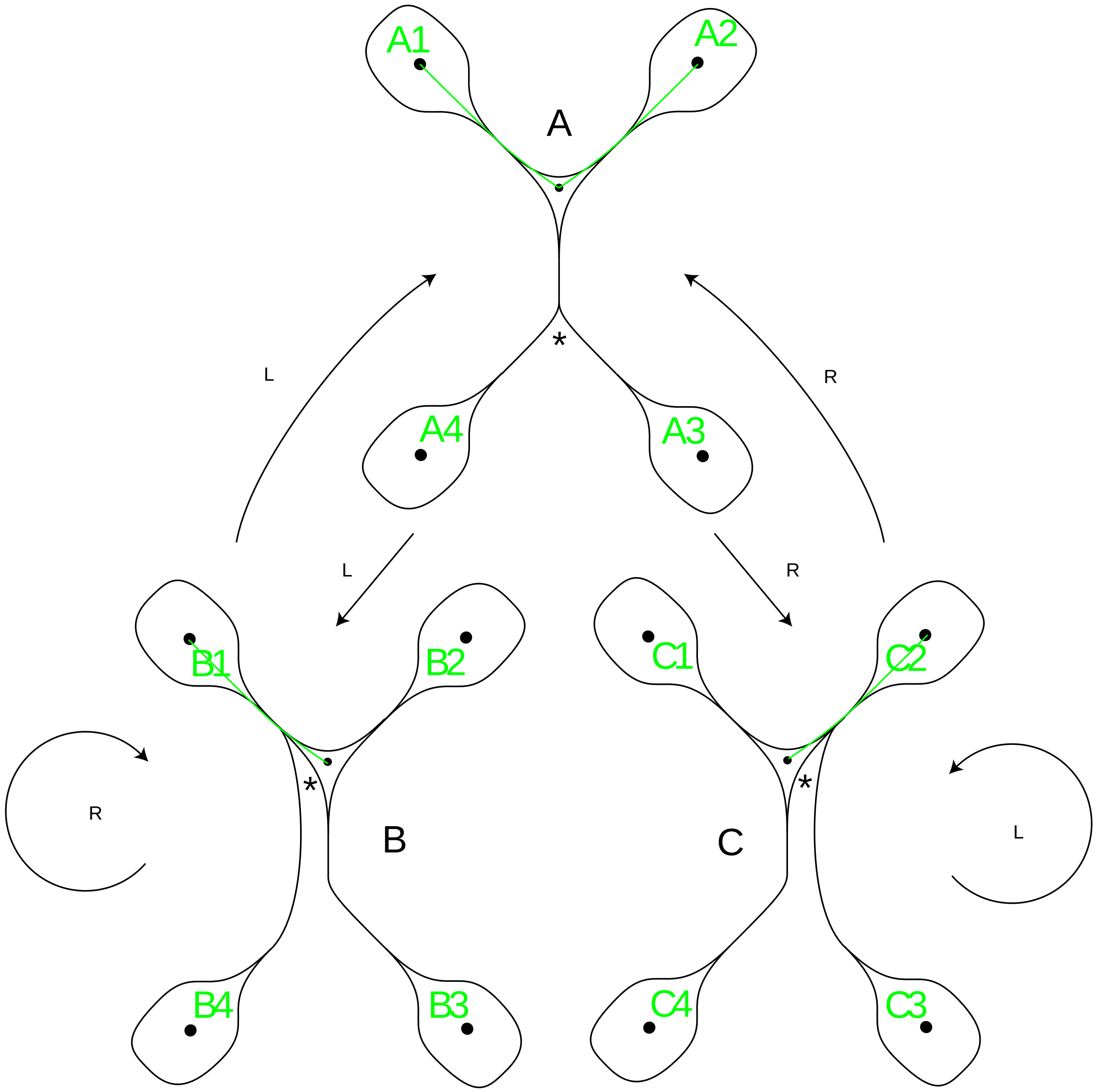

FIGURE 3. Various splittings

If they don't then the lamination is arational and the corresponding homeomorphism is pseudo-Anosov.

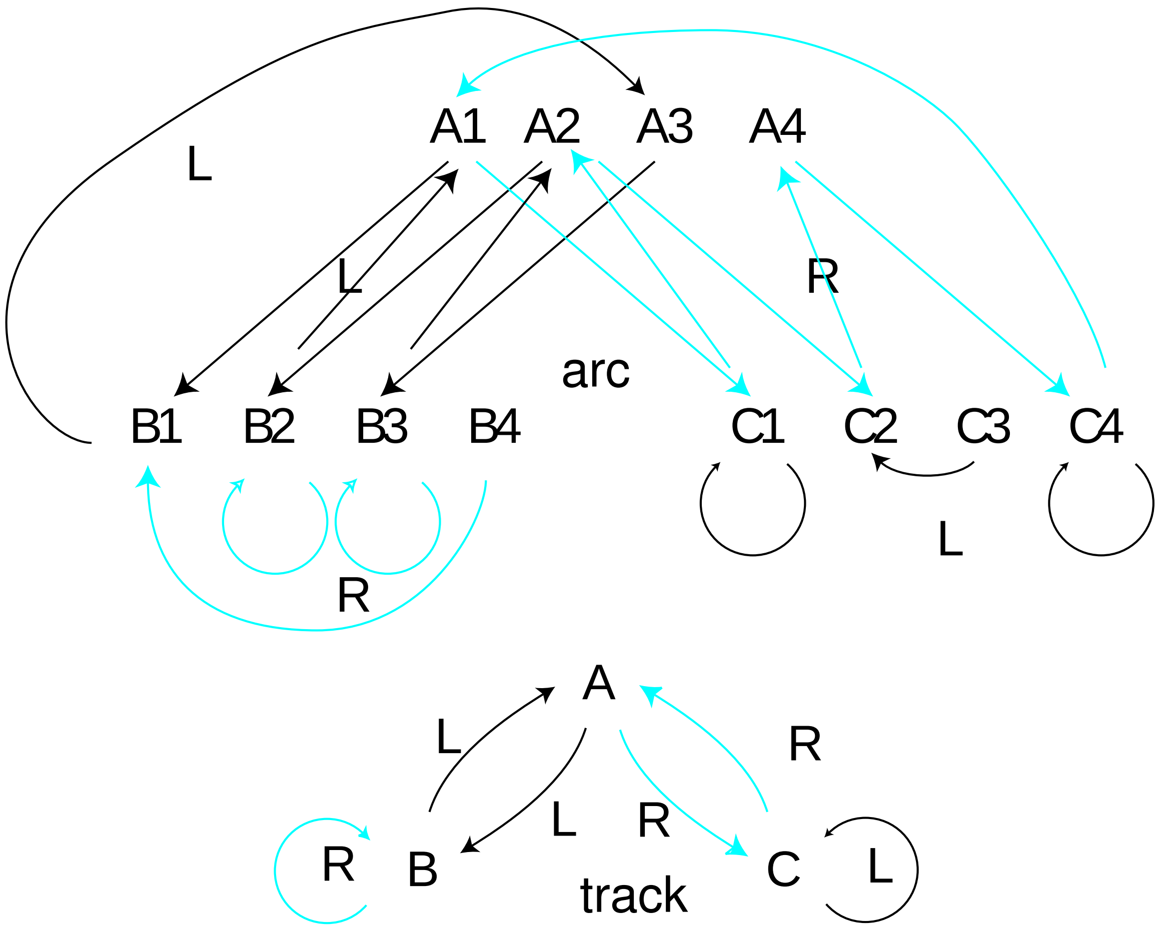

FIGURE 4. The arc and train track automata