# CIRCLE LINKS

The Borromean rings have the property that removing one of the components leaves the unlink. Since each component is unknotted, and an unknot may be represented by a round circle, can one make the Borromean rings out of round circles? It is not too hard to show that the Borromean rings may be made out of ellipses which are arbitrarily close to circles. To see this, take three orthogonal planes, and three congruent non-circular ellipses lying in the orthogonal planes with centers at the intersection, such that the major and minor axes of the ellipses lie in the intersections between pairs of planes (see figure 1).

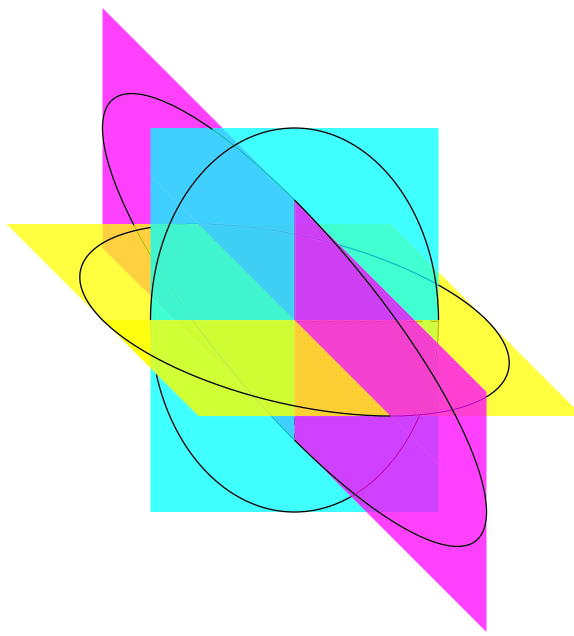

Figure 1. Ellipse Borromean rings

If one chooses these so that the ellipses are disjoint, then the union of the three ellipses form the Borromean rings. Letting the major and minor radii approach each other, one obtains Borromean rings with the components arbitrarily close to circles. Michael Freedman has shown that one cannot make the components simultaneously circular. More generally, if one has a link for which every pair of components have linking number 0, then the link is the unlink: it may be isotoped so that each component is a circle which bounds a disk disjoint from all the other components. This proof is an example of a "book proof", a proof which makes the theorem transparent. Erdös believed that God has a book with the most elegant proofs of every theorem written in it, and one of mathematicians' goals is to discover what is written in this book.

Freedman's proof uses the fact that $R^3$ is a subspace of $\mathbb{R}^4$. In order to understand 4-dimensions, it is sometimes easier to visualize a 2-dimensional analog which has some of the essential features of the 4-dimensional setup, a technique that Freedman is expert at.

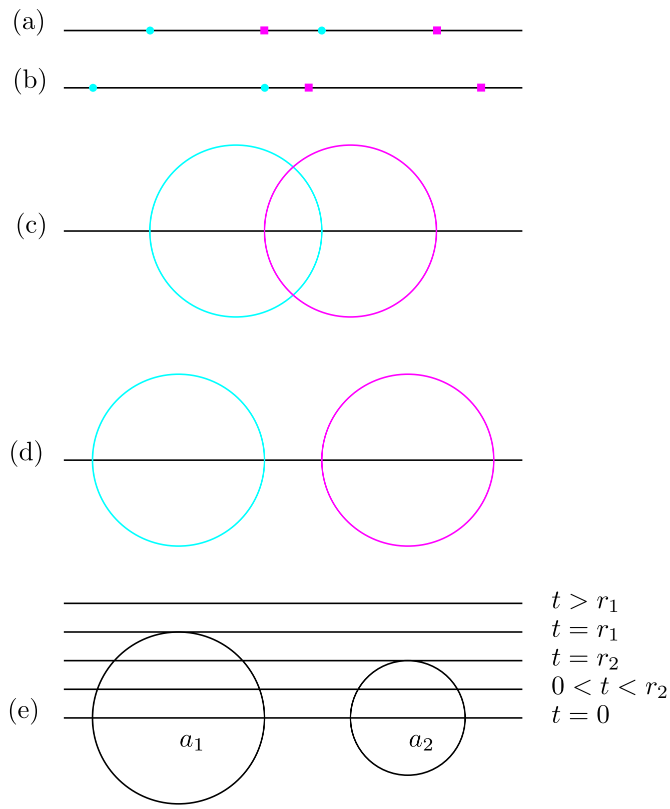


FIGURE 2. Shrinking unlinked $S^0$s

Take an embedding of disjoint $S^0$'s in the line. Each $S^0$ consists of two points, a solution of the equation $|x - a| = r$. If we consider $\mathbb{R}^1 \subset \mathbb{R}^2$, then each $S^0$ extends to a circle $(x-a)^2 + y^2 = r^2$. Two $S^0$'s are *linked* if and only if they are interleaved in $\mathbb{R}$, that is, one pair of points separates the other pair (see figure 2(a) for linked $S^0$s and 2(b) for unlinked $S^0$s). If the two $S^0$s are defined by $|x - a_i| = r_i$, $i = 1, 2$, then the corresponding circles will intersect if and only if the $S^0$s are linked (see 2(c) and (d)). If we have $n$ $S^0$s embedded in $\mathbb{R}$, so that each pair is unlinked, then all of the corresponding circles in $\mathbb{R}^2$ will be disjoint. Now, we move the line $y = 0$ upward, as a 1-parameter family of lines $y = t$, $t \geq 0$. For a given circle $(x-a)^2 + y^2 = r^2$, the line $y = t$ intersects it in two points for $|t| < r$, one point when $|t| = r$, and no points when $|t| > r$. Thus, if we look at the collection of circles corresponding to our $S^0$s, we see each of the 0-spheres shrink down to a point, then disappear.

We may perform a similar operation for $\mathbb{R}^3$ thought of as lying in $\mathbb{R}^4$. If we have a circle in $\mathbb{R}^3$, then there is a unique 2-sphere in $\mathbb{R}^4$ which is perpendicular to $\mathbb{R}^3 \subset \mathbb{R}^4$, and intersects $\mathbb{R}^3$ in the circle. The linking number of circles is a bit trickier to define than for $\mathbb{R}$, but in the case of circles it is straightforward to describe. Given two circles, take a disk bounding one of the circles. Then the two circles are linked if this disk intersects the other one in exactly one point (figure 3(c)), and are unlinked otherwise (3(a),(b))(the disk and circle can intersect in at most two points, so they are unlinked if they intersect in 0 or 2 points). One may see that the 2-spheres in $\mathbb{R}^4$ corresponding to two circles in $\mathbb{R}^3$ intersect if and only if the circles are linked (this follows from the 2-dimensional case, by taking all 2-planes in $\mathbb{R}^4$ perpendicular to $\mathbb{R}^3$, and observing that the intersection looks like $S^0$s in $\mathbb{R} \subset \mathbb{R}^2$). Thus, if we have $n$ circles such that all pairs are unlinked, then the corresponding

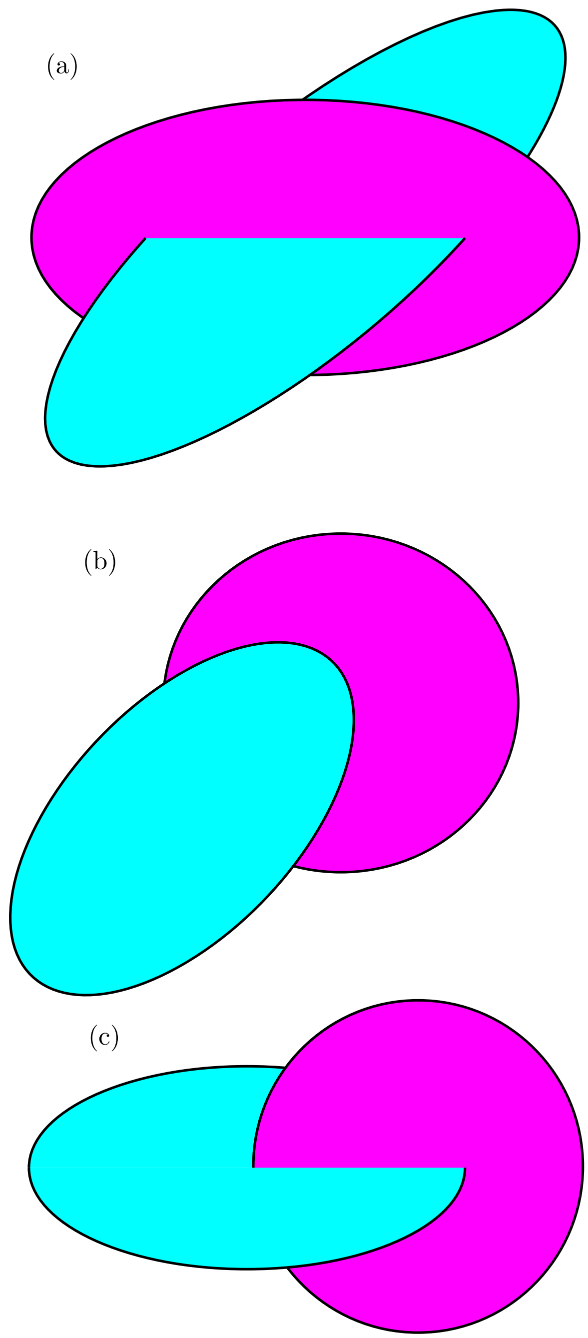


FIGURE 3. (a),(b) Unlinked, (c) linked circles in $\mathbb{R}^3$

2-spheres in $\mathbb{R}^4$ will be disjoint. Now, look at parallel copies of $\mathbb{R}^3$ in $\mathbb{R}^4$ at distance $t$ from our initial $\mathbb{R}^3$. If we consider the intersection with a perpendicular sphere of radius $r$, then we see a family of circles of radii $\sqrt{r^2 - t^2}$ shrinking to a point at time $t = r$. When we look at the movie of the entire link, we see all of the circles shrinking to points in finite time, then disappearing. This shows that our link is isotopic to the unlink. This is one point where the analogy of $\mathbb{R}^3$ with $\mathbb{R}$ fails. In the 3-dimensional case, once each circle has nearly shrunk to a point, we may move it far away from the others (say, separated by a hyperplane from the rest of the link). Then, we may see that we have the unlink with $n$ components, proving Freedman's claim. One cannot do this in $\mathbb{R}^1$: once an $S^0$ has shrunk to a point, we can't move it away from the others.

The classification of circle links without restrictions on the linking number is much more subtle in general, and might make for an interesting research project.