

Maple Lecture 3. Exact and Floating-Point Numbers

The ability to compute exactly or with high precision is one of the main features of computer algebra. The material in this lecture is based on [2, sections 2.3 and 2.4]. See also [4]. For insight in multiprecision arithmetic, we refer to [1, 3, 5].

3.1 Integers and Rational Numbers

Maple can handle very large integer numbers:

```
[> n1 := 3^(4^5);
```

Observe that the placing of the brackets matter:

```
[> n2 := (3^4)^5;
```

The largest integer number in MapleV had $2^{19} - 9$ digits. Since Maple 7, this limit was raised a bit:

```
[> md := kernelopts(maxdigits); # query a kernel variable
[> huge := 10^md;                # may take time, depends on available memory
[> length(huge);                 # show number of digits
```

Note that we do not really wish to see this number; therefore we close off with a colon, instead of a semicolon. Let us try to push the limits:

```
[> huger := 10*huge;
```

Note that limitations of available memory may set the limit on the largest number much lower than indicated by the kernel variable `maxdigits`. Since Maple 9, the GNU Multiple Precision (GMP) library is used for its long integer arithmetic.

You may wonder what these huge numbers are good for. In computational number theory, with the applications in coding and cryptography, prime numbers play an important role. For example the public key RSA cryptosystem is based on the difficulty of factoring integer number into primes.

Maple has fast algorithms to check whether a number is prime or not:

```
[> n1 := 2^301-1;                # a very large number
[> isprime(n1);                  # this is not a prime, but we
[> n2 := nextprime(n1);          # can get the next prime number
[> ifactor(n2);                  # see if the number of prime
[> ifactor(n1);                  # this takes a long time...
```

While probabilistic algorithms exist to decide fast whether a number is prime or not, to factor a number in primes, one essentially has to try as many factors as the integer square root of the number. In this case the number of steps for this factorization is of the order of

```
[> isqrt(n1);
```

Notice the little ‘i’ at the start of the commands `ifactor` and `isqrt` to indicate that they operate on integers and return integer results. Another important operation is the greatest common divisor of two integer numbers. In the assignments we will explore the capabilities of the `igcdex` command, which solves the extended gcd problem.

Maple automatically simplifies rational numbers so that numerator and denominator are relatively prime to each other:

```
[> a := 1234/4890;               # numerator and denominator are even
[> gcd(numer(a),denom(a));       # shows simplification
```

3.2 Irrational and Floating-Point Numbers

Not all numbers have an exact representation. To get an idea of the actual value of such numbers we approximate using floating-point numbers.

By default, Maple shows 10 *significant* decimal places, as the following experiment shows:

```
[> evalf(Pi);
[> evalf(Pi*1000);
[> evalf(Pi/1000);
```

So leading zeros are not counted as significant digits.

The number of digits determines the working precision of the floating-point calculations in Maple, and in particular the size of the roundoff, for example:

```
[> r := evalf(5^(1/3));
[> r^3;
```

As `r` is a floating-point approximation of the real cube root of 5, we expected to see 5 as the result of the last command, but the result was spoiled by roundoff. To reduce the roundoff we use more decimal places. We can do this in two ways: either adding the number of decimal places to the argument of `evalf`, or assigning a higher value to the variable `Digits`.

The floating-point arithmetic is in general software driven, except in cases where Maple can deduce the needed working precision for the value of `Digits`. Try this:

```
[> UseHardwareFloats;
```

Like `Digits`, `UseHardwareFloats` is another environment variable which can be set by us to the values `true`, `false`, or `deduced`. Concerning the last value, hardware floating-point arithmetic can be performed when `Digits <= evalhf(Digits)`.

In applications where 14 or fewer decimal places suffice, such as in plotting a function which is easy to evaluate, hardware floating-point arithmetic is usually much faster. With `evalhf`, we get access to the machine arithmetic.

With the following experiment we illustrate the relevance of using hardware floats. The piece of code given below could be part of sampling the exponential function for plotting purposes.

```
[> N := 10000:           # number of samples
[> p := 20:              # required precision
```

Depending on the actual speed of our machine, we can increase the values for `N` and `p` to experience more clearly the difference between software and hardware floating-point calculations.

```
[> start_time := time():      # start recording CPU time
[> for i from 1 to N do      # loop from 1 to N
[>   a := evalf(exp(i),p):    # take samples from the function exp
[> end do:                  # terminate the loop
[> elapsed := time()-start_time; # display elapsed CPU time
```

Now we redo the calculation with hardware floats.

```
[> start_time := time():
[> for i from 1 to N do
[>   a := evalhf(exp(i)):
[> od:
[> elapsed := time()-start_time;
```

Observe the differences in elapsed times between the software driven and hardware machine arithmetic.

3.3 Assignments

1. What happens when you type in 3^4^5 ; in a Maple session? Why does Maple react in this way?
(*Hint*: consult `?operators[precedence]`.)
2. Write $5^4^3 - 1$ as a product of primes.
3. The greatest common divisor of two integer numbers a and b can be written as a linear combination (with integer coefficients k and l) of a and b : $\gcd(a, b) = ka + lb$.
In Maple this is achieved with the command `igcdex`. Look in the help page of this command to compute the coefficients of the linear combination of the greatest common divisor of 12214 and 2012. Give the command you type in to find these coefficients and also give the command(s) to verify the result.
4. In Maple, what is the difference between $1/3 + 1/3 + 1/3$ and $1/3.0 + 1/3.0 + 1/3.0$?
5. Consecutive rational approximations for π are $\frac{22}{7}$, $\frac{333}{106}$, $\frac{355}{113}$, $\frac{104348}{33215}$, \dots
What is the next more accurate rational approximation for π ?
(*Hint*: search the Maple Help how to convert a float to an approximate rational number.)
6. Calculate $e^{\sqrt{\pi}}$ with 10, 20, and 30 decimal places accurately.
7. When calculating in high precision, we may want to restrict the output. The number of displayed decimal places is controlled by the interface variable `displayprecision` (consult `?displayprecision`).
Set the display to four decimal places and execute `sin(evalf(Pi))`. Explain how you can see from the output of this command that the working precision is more than four decimal places.
8. Explain the difference between `1.0+10^(-20)` and `1+10^(-20)` in Maple.
How can you make Maple return the same *correct* value of these two sums?
9. Set up an experiment to get an idea about the cost of working with large number as the digits rise.
Consider the example of evaluating the exponential function N times at π for increasing precision p .
Do the following calculations:
 - (a) Start with $p = 10$ and double the precision 7 times.
 - (b) Start with $p = 20$ and increase the precision 7 times with 20.
 For both calculations, display the elapsed CPU time in a table.
10. By default, the working precision is 10 decimal places. Find how to change this default globally, without using the `Digits` command. (*Hint*: on Windows, look under Tools/Options; on Unix, look under File/Preferences; on Mac, look under Maple 12/Preferences.) Set it to the more reasonable value returned by `evalhf(Digits)`.

References

- [1] K.O. Geddes, S.R. Czapor, and G. Labahn. *Algorithms for Computer Algebra*. Kluwer Academic Publishers, 1992.
- [2] A. Heck. *Introduction to Maple*. Springer-Verlag, third edition, 2003.
- [3] D.E. Knuth. *The Art of Computer Programming. Volume 2. Seminumerical Algorithms*. Addison-Wesley, third edition, 1998.
- [4] D.I. Schwartz. *Introduction to Maple 8*. Prentice-Hall, 2003.
- [5] J. von zur Gathen and J. Gerhard. *Modern Computer Algebra*. Cambridge University Press, 1999.