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# On Computing Characteristic Sets of Arbitrary Radical Differential Ideals

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Abstract. In this paper we study the problem of computing a Kolchin characteristic set of a radical differential ideal. The central part of the article is the presentation of an algorithm that solves this problem in the case of ordinary differential polynomials and orderly rankings. We also discuss the usefulness of regular and characteristic decompositions of radical differential ideals and the problem of computation of characteristic sets in the case of partial differential polynomials. In the partial differential case we give an algorithm for computing characteristic sets in the special case of radical differential ideals satisfying the property of consistency.

## 1. Introduction

This paper is devoted to study radical differential ideals and their characteristic sets. The concept of a characteristic set introduced by Ritt and Kolchin is one of the most important notions in differential algebra. The problem of computing a characteristic set of a radical differential ideal represented by a finite set of its generators is not completely solved yet especially in the partial differential case. In the case of ideals in rings of polynomials in a finite number of variables this problem was studied and completely solved by Gallo and Mishra [**6, 7, 8**]. It was also investigated by Aubry, Lazard and Moreno Maza [**2**].

So, it is very natural to study this problem in rings of differential polynomials. The most important contributions of this article are:

- an algorithm for computing characteristic sets of arbitrary radical differential ideals w.r.t. orderly rankings in the ordinary case;
- an algorithm for computing characteristic sets of radical differential ideals satisfying the property of consistency w.r.t. orderly rankings in the partial differential case.

We also conjecture a method of soving this problem in non-ordinary cases for arbitrary radical differential ideals (see Section 6).

We use other technique and methods than that used by Gallo and Mishra. However, their algorithm for computing a characteristic set of an algebraic ideal

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plays an important role in Algorithm 1 and Algorithm 2 of this paper. The methods developed by Sadik [**17**] help us to obtain several bounds for characteristic sets w.r.t. different rankings. Thus, the results of this paper (Theorem 4 and Theorem 5) are new in comparison with Sadik's ones.

Ten years ago a technique for effective and factorization-free computations in the radical differential ideal theory was developed by Boulier, Lazard, Ollivier and Petitot (see [**4**] and [**5**]). In [**9, 10**] Hubert continued to develop this problem and introduced the notions of *characterizable* ideal and *characteristic decomposition* of a radical differential ideal. This decomposition of the ideal helps us to solve many problems concerning the system of differential equations associated with the ideal and to test the radical membership.

It should be emphasized that a characteristic decomposition of a radical differential ideal does not give us full information about the ideal. In some important cases a representation of this ideal by characteristic components cannot replace a representation of the ideal by its generators as a radical differential ideal. For example, at this moment one cannot check the inclusion of a radical differential ideal to another radical differential ideal knowing only a characteristic decomposition of the first one (see [**12, 14, 15**]). This problem is closely related to the well-known Ritt problem. In this case it is necessary to know generators of the ideal and characteristic decomposition is partially useless.

Nevertheless, in this paper we show that a characteristic decomposition of a radical differential ideal knows a lot about the ideal. Indeed, characteristic decomposition allows us not only to test membership to this ideal but also to compute its characteristic set in Kolchin's sense. Hence, the main contribution of this paper can be also considered as another application of a characteristic decomposition. Such a decomposition tells us a lot about a system of partial differential equations and we study what else one can do using a characteristic decomposition.

We also study the case of *partial* differential polynomials and give an algorithm (Algorithm 2) for computing a Kolchin characteristic set. Theorem 5 in Section 3 provides a theoretical basis for it. Radical differential ideals, for which we propose the algorithm, satisfy the property of consistency (see Definition 4) w.r.t. an orderly ranking. We will see that these ideals are *better* than arbitrary radical differential ideals in the *computational sense*.

In summary, although a characteristic decomposition cannot replace the ideal in computational sense, this decomposition allows us to compute such an important subset of a radical differential ideal as its characteristic set.

## 2. Preliminaries

**2.1. Basic definitions.** Differential algebra deals with differential rings and fields. These are commutative domains with 1 and a basic set of differentiations $\Delta = \{\delta_1, \ldots, \delta_n\}$ on a ring. The case of $\Delta = \{\delta\}$ is called *ordinary.* If $R$ is an ordinary differential ring and $y \in R$ we denote $\delta^k y$ by $y^{(k)}$. The ring of differential polynomials was introduced to deal with algebraic differential equations.

Recent tutorials on constructive differential ideal theory are presented in [**10, 18**]. We also use the Gröbner bases technique discussed in detail in [**3**]. The definition of the ring of *differential polynomials* in $l$ variables over a differential field $k$ is given in [**11, 13, 16**]. This ring is denoted by $k\{y_1, \ldots, y_l\}$. We consider the case of $\operatorname{char} k = 0$ only. We denote polynomials by $f, g, h, \ldots$ and use the notation $I, J, P, Q$ for ideals.

We need the notion of reduction for algorithmic computations. First, we introduce a *ranking* on the set of differential variables of $k\{y_1, \ldots, y_l\}$. Construct the multiplicative monoid $\Theta = (\delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_n}, k_i \geqslant 0)$. The ranking is a total ordering on the set $\{\theta y_i\}$ for each $\theta \in \Theta$ and $1 \leqslant i \leqslant l$ satisfying the following conditions:

$$\theta u \geqslant u, \quad u \geqslant v \Longrightarrow \theta u \geqslant \theta v.$$

In later discussions we suppose that a ranking is fixed.

Let $u$ be a differential variable in $k\{y_1, \ldots, y_l\}$, that is, $u = \theta y_j$ for a differential operator $\theta = \delta_1^{k_1}\delta_2^{k_2}\cdots\delta_n^{k_n} \in \Theta$ and $1 \leqslant j \leqslant l$. Set $\operatorname{ord}_w u = \sum_{i=1}^{n} w_i k_i$, where $w_i$ are positive integers for $1 \leqslant i \leqslant n$ and $w = (w_1, \ldots, w_n)$. From now we suppose that some $w$ is fixed and denote $\operatorname{ord}_w$ simply by ord. A ranking is said to be *orderly* iff $\operatorname{ord} u > \operatorname{ord} v$ implies $u > v$ for all differential variables $u$ and $v$. A ranking $>_{el}$ is called *elimination* iff $y_i >_{el} y_j$ implies $\theta_1 y_i >_{el} \theta_2 y_j$ for all $\theta_1, \theta_2 \in \Theta$.

The highest ranked derivative $\theta y_j$ appeared in a differential polynomial $f \in k\{y_1, \ldots, y_l\} \setminus k$ is called the leader of $f$. We denote the leader by $u_f$. Represent $f$ as a univariate polynomial in $u_f$:

$$f = I_f u_f^n + a_1 u_f^{n-1} + \ldots + a_n.$$

The polynomial $I_f$ is called the *initial* of $f$.

Apply any $\delta \in \Theta$ to $f$:

$$\delta f = \frac{\partial f}{\partial u_f}\delta u_f + \delta I_f u_f^n + \delta a_1 u_f^{n-1} + \ldots + \delta a_n.$$

The leading variable of $\delta f$ is $\delta u_f$ and the initial of $\delta f$ is called the *separant* of $f$. We denote it by $S_f$. Note that for all $\theta \in \Theta$, $\theta \neq 1$, each $\theta f$ has the initial equal to $S_f$.

Define the ranking on differential polynomials. We say that $f > g$ iff $u_f > u_g$ or in the case of $u_f = u_g$ we have $\deg_{u_g} f > \deg_{u_g} g$. Let $F \subset k\{y_1, \ldots, y_l\}$ be a set of differential polynomials. For the differential and radical differential ideal generated by $F$ in $k\{y_1, \ldots, y_l\}$, we use the notation $[F]$ and $\{F\}$, respectively.

We say that a differential polynomial $f$ is *partially reduced* w.r.t. $g$ iff no proper derivative of $u_g$ appears in $f$. A differential polynomial $f$ is *reduced* w.r.t. $g$ iff $f$ is partially reduced w.r.t. $g$ and $\deg_{u_g} f < \deg_{u_g} g$. Consider any subset $\mathbb{A} \subset k\{y_1, \ldots, y_l\}$. We say that $\mathbb{A}$ is autoreduced iff $\mathbb{A} \cap k = \varnothing$ and each element of $\mathbb{A}$ is reduced w.r.t. all the others. It is proved in [**11**] that every autoreduced set is finite. For autoreduced sets we use capital letters $\mathbb{A}, \mathbb{B}, \mathbb{C}, \ldots$.

We denote the product of the initials and the separants of the elements of $\mathbb{A}$ by $I_{\mathbb{A}}$ and $S_{\mathbb{A}}$, respectively. Denote $I_{\mathbb{A}} \cdot S_{\mathbb{A}}$ by $H_{\mathbb{A}}$. Let $S$ be a finite set of differential polynomials. Denote by $S^\infty$ the multiplicative set containing 1 and generated by $S$. Let $I$ be an ideal in a commutative ring $R$. Let $I : S^\infty = \{a \in R | \exists s \in S^\infty : sa \in I\}$. If $I$ is a differential ideal then $I : S^\infty$ is also a differential ideal (see [**11, 16, 13, 18**]).

If we want to enumerate the elements of $\mathbb{A}$ we write the following: $\mathbb{A} = A_1, A_2, \ldots, A_p$. Let $\mathbb{A} = A_1, \ldots, A_r$ and $\mathbb{B} = B_1, \ldots, B_s$ be autoreduced sets. Let the elements of $\mathbb{A}$ and $\mathbb{B}$ be arranged in order of increasing rank. We say that $\mathbb{A}$ has lower rank than $\mathbb{B}$ iff there exists $k \leqslant r, s$ such that $\operatorname{rank} A_i = \operatorname{rank} B_i$ for

$1 \leqslant i < k$ and $\operatorname{rank} A_k < \operatorname{rank} B_k$, or if $r > s$ and $\operatorname{rank} A_i = \operatorname{rank} B_i$ for $1 \leqslant i \leqslant s$. We say that $\operatorname{rank} \mathbb{A} = \operatorname{rank} \mathbb{B}$ iff $r = s$ and $\operatorname{rank} A_i = \operatorname{rank} B_i$ for $1 \leqslant i \leqslant r$.

Consider two differential polynomials $f$ and $g$ in $R = k\{y_1, \ldots, y_l\}$. Let $I$ be the differential ideal in $R$ generated by $g$. Applying a finite number of differentiations and pseudo-divisions one can compute a *differential partial remainder* $f_1$ and a *differential remainder* $f_2$ of $f$ w.r.t. $g$ such that there exist $s \in S_g^\infty$ and $h \in H_g^\infty$ satisfying $sf \equiv f_1 \mod I$ and $hf \equiv f_2 \mod I$ with $f_1$ and $f_2$ partially reduced and reduced w.r.t. $g$, respectively (see [**9**] for defintions and an algorithm for computing remainders).

Let $\mathbb{A}$ be an autoreduced set in $k\{y_1, \ldots, y_l\}$. Consider the polynomial ring $k[x_1, \ldots, x_n]$ with $x_1, \ldots, x_n$ belong to $\Theta Y$ for $Y = y_1, \ldots, y_l$. Let $U, V \subset \{x_1, \ldots, x_n\}$ be the sets of "leaders" and "non-leaders" appearing in the autoreduced set $\mathbb{A}$, respectively. We denote $k[x_1, \ldots, x_n]$ by $k[V][U]$ and the leader of $A_i$ by $u_{A_i}$ or $u_i$ for each $1 \leqslant i \leqslant p$.

Example 1. Let $\mathbb{A} = A_1, A_2 \subset k\{v, u_1, u_2\}$, where $A_1 = vu_1^2 + u_1 + v^2, A_2 = u_1u_2^3 + v$ and $v < u_1 < u_2$. We have $U = u_1, u_2$ and $V = v$.

The notion of a characteristic set in *Kolchin's sense* in characteristic zero is important in our further discussions.

Definition 1. [**11**, page 82] An autoreduced set of the lowest rank in an ideal $I$ is called a *characteristic set* of $I$.

As it is mentioned in [**11**, Lemma 8, page 82], in characteristic zero $\mathbb{A}$ is a characteristic set of a proper differential ideal $I$ iff each element of $I$ reduces to zero w.r.t. $\mathbb{A}$. Consider the definition of a characterizable radical differential ideal.

Definition 2. [**9**, Definition 2.6] A radical differential ideal $I$ in $k\{y_1, \ldots, y_l\}$ is said to be *characterizable* iff there exists a characteristic set $\mathbb{A}$ of $I$ in Kolchin's sense such that $I = [\mathbb{A}] : H_{\mathbb{A}}^\infty$.

The following definition makes a bridge between differential and commutative algebra. Let $v$ be a derivative in $k\{y_1, \ldots, y_l\}$. $\mathbb{A}_v$ is the set of the elements of $\mathbb{A}$ and their derivatives that have a leader ranking strictly lower than $v$.

Definition 3. [**11**, III.8] $\mathbb{A}$ is *coherent* iff whenever $A, B \in \mathbb{A}$ are such that $u_A$ and $u_B$ have a common derivative: $v = \psi u_A = \phi u_B$, then $S_B\psi A - S_A\phi B \in (\mathbb{A}_v) : H_{\mathbb{A}}^\infty$.

We emphasize that a characteristic set of a differential ideal is a coherent autoreduced set (see [**11, 16, 13, 18**]).

**2.2. Important assertions.** Consider several important results concerning radical differential ideals in rings of differential polynomials. The technique described in [**9, 11**] helps us to cover some properties of these ideals.

Theorem 1. [**11**, III.8, Lemma 5] *Let $\mathbb{A}$ be a coherent autoreduced set in $k\{y_1, \ldots, y_l\}$. Suppose that a differential polynomial $g$ is partially reduced w.r.t. $\mathbb{A}$. Then $g \in [\mathbb{A}] : H_{\mathbb{A}}^\infty$ iff $g \in (\mathbb{A}) : H_{\mathbb{A}}^\infty$.*

Note that Theorem 1 is also known as *Rosenfeld's lemma*.

Theorem 2. [**9**, Theorem 3.2] *Let $\mathbb{A}$ be an autoreduced set of $k[V][U]$. If $1 \notin (\mathbb{A}) : S_{\mathbb{A}}^\infty$ then any minimal prime of $(\mathbb{A}) : S_{\mathbb{A}}^\infty$ admits the set of non-leaders of*

*$\mathbb{A}$, $V$, as a transcendence basis. More specially, any characteristic set of a minimal prime of $(\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ has the same set of leaders as $\mathbb{A}$.*

THEOREM 3. [**9**, Theorem 4.5] *Let $\mathbb{A}$ be a coherent autoreduced set of $R = k\{y_1, \dots, y_l\}$ such that $1 \notin [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. There is a one-to-one correspondence between the minimal primes of $(\mathbb{A}) : H_{\mathbb{A}}^{\infty}$ in $k[V][U]$ and the essential prime components of $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ in $R$. Assume $\mathbb{C}_i$ is a characteristic set of a minimal prime of $(\mathbb{A}) : H_{\mathbb{A}}^{\infty}$. Then $\mathbb{C}_i$ is the characteristic set of a single essential prime component of $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ (and vice versa).*

LEMMA 1. *Let $\mathbb{A} = A_1, \dots, A_p$ be an autoreduced set in the ring $k[x_1, \dots, x_m] = R$ and a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$. Suppose that a polynomial $f = a_m x_t^m + \dots + a_0 \in R$ is reducible to zero w.r.t. $\mathbb{A}$ and the indeterminate $x_t$ does not appear in $A_i$ for each $1 \leqslant i \leqslant p$. Then $a_j$ is reducible to zero w.r.t. $\mathbb{A}$ for all $0 \leqslant j \leqslant m$.*

PROOF. Since $f$ is reducible to zero w.r.t. $\mathbb{A}$, there exists $I \in I_{\mathbb{A}}^{\infty}$ such that

$$I \cdot f = \sum_{i=1}^{p} g_i A_i.$$

Let $g_i = \sum\limits_{j=1}^{t_i} h_j x_t^j$ for each $1 \leqslant i \leqslant p$. Thus, we have $I \cdot \sum\limits_{k=0}^{m} a_k x_t^k = \sum\limits_{k=0}^{q} d_k x_t^k$ with $d_k \in (A_1, \dots, A_p)$. Hence, $I \cdot a_i \in (\mathbb{A})$ for each $1 \leqslant i \leqslant m$, that is, $a_i \in (\mathbb{A}) : I_{\mathbb{A}}^{\infty}$. Since $\mathbb{A}$ is a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$, we have all $a_i$ are reducible to zero w.r.t. $\mathbb{A}$. □

## 3. The ordinary case

**3.1. Bounds for orders of characteristic sets.** Denote [**9**, Algorithm 7.1] by $\chi$-Decomposition. An input of this algorithm is a finite set of differential polynomials $F$ and its output is a set $\mathfrak{C} = \mathbb{C}_1, \dots, \mathbb{C}_n$ of characteristic sets $\mathbb{C}_i$ of characterizable ideals $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ forming a characteristic decomposition of the radical differential ideal $\{F\}$ in $k\{y_1, \dots, y_l\}$:

$$\{F\} = [\mathbb{C}_1] : H_{\mathbb{C}_1}^{\infty} \cap \dots \cap [\mathbb{C}_n] : H_{\mathbb{C}_n}^{\infty}.$$

We will show how this decomposition helps to compute a characteristic set of $\{F\}$.

The main idea of Algorithm 1 is to move our problem into commutative algebra. In order to do this we need to know bounds for the orders of elements of a characteristic set of a radical differential ideal.

Let $R = k\{y_1, \dots, y_l\}$ with $\Delta = \{\delta\}$. So, we are in the ordinary case. *Differential dimension* of a prime differential ideal $P$ is the maximal number $q$ such that $P \cap k\{y_{i_1}, \dots, y_{i_q}\} = \{0\}$. If $f$ is a differential polynomial then $\operatorname{ord} f$ denotes the maximal order of differential variables appeared effectively in $f$. Let $\mathbb{A} = A_1, \dots, A_p$ be an autoreduced set. Define the order of $\mathbb{A}$ by the following equality: $\operatorname{ord} \mathbb{A} = \operatorname{ord} A_1 + \dots + \operatorname{ord} A_p$. If a set $\mathbb{C}$ is characteristic of the ideal $P$ w.r.t. an orderly ranking then by definition the order of the ideal $P$ equals $\operatorname{ord} \mathbb{C}$.

REMARK 1. Almost all recent result in the dimensional theory for prime differential ideals (the theory of differential dimension polynomials) are presented in [**13**].

LEMMA 2. [**17**, Proposition 17] *Consider a prime differential ideal $P$ of differential dimension $q$ and of order $h$. For every subset $\{y_{i_1},\dots,y_{i_{q+1}}\}$ of $\{y_1,\dots,y_l\}$ the ideal $P$ contains a differential polynomial in the indeterminates $\{y_{i_1},\dots,y_{i_{q+1}}\}$ with order less than or equal to $h$.*

LEMMA 3. [**17**, Lemma 23] *Let $P$ be a prime differential ideal, then $P$ admits a characteristic set $\mathbb{C} = C_1,\dots,C_p$ such that $\dfrac{\partial C_i}{\partial v_k^{(l_{i,k})}}$ does not lie in $P$, where $v_k$ is a non-leader and $l_{i,k} = \mathrm{ord}(\mathbb{C}_i, v_k)$ for a pair $(i,k)$ in $\{1,\dots,p\}\times\{1,\dots,n-p\}$.*

LEMMA 4. *Let $P$ be a prime differential ideal in $k\{y_1,\dots,y_l\}$ and $>$ be a differential ranking on the set of differential variables $Y = \{y_1,\dots,y_t\}$ for some fixed $t$, $1 \leqslant t < p$, where $p$ is the number of elements of a characteristic set of the ideal $P$ w.r.t. an orderly ranking. Let the order of a characteristic set of $P$ be equal to $h$. Then the order in $y_{t+1}$ of each element of a characteristic set $\mathbb{C} = C_1,\dots,C_n$ of $P$ w.r.t. the "mixed" ranking $Y >_{el} y_{t+1} >_{el} \dots >_{el} y_l$ does not exceed $h$.*

PROOF. The proof of [**17**, Theorem 24] will be valid for our purpose if we modify it in the following way. If for some $\theta \in \Theta$ the variable $\theta y_{t+1}$ is the leader of some $C_j$ then $\mathrm{ord}(C_q, y_{t+1}) \leqslant h$ for all $1 \leqslant q \leqslant n$ immediately, since $\mathbb{C}$ is autoreduced. Let $\theta y_{t+1}$ be a non-leader for all $\theta\in\Theta$.

The main idea is to reduce the polynomial $f_{C_j}(y_j, y_{t+1},\dots,y_l)$ given by Lemma 2 w.r.t. $\mathbb{C}$. Suppose that $\mathrm{ord}(C_q, y_{t+1}) > h$ and $\mathrm{ord}(f, y_q) > \mathrm{ord}(C_q, y_q)$. The case of $\mathrm{ord}(f, y_q) = \mathrm{ord}(C_q, y_q)$ is the same as in the proof of [**17**, Theorem 24].

Our improvement is to choose $C_i = \arg\max\limits_{C_j\in\mathbb{C}} \mathrm{ord}(C_j, y_{t+1})$ instead of using induction as Sadik did. Let $u_i = \theta_i y_i$ for some $\theta_i \in \Theta$ and $u_i$ is the leader of $C_i$. We have $s = \mathrm{ord}(C_i, y_{t+1}) > h$ and $r_f = \mathrm{ord}(f_{C_i}, y_i) > \mathrm{ord}(C_i, y_i) = r_C$, where $f_{C_i} = f_{C_i}(y_i, y_{t+1},\dots,y_l)$. Let us reduce $f_{C_i}$ first by $C_i$. We need to differentiate $C_i$ not more than $r_C - r_f$ times and get the remainder $\tilde f$.

It should be emphasized that applying further steps of reduction to $\tilde f$ by other $C_j$ we need to differentiate them less than $r_C - r_f$ times, since $\mathbb{C}$ is autoreduced. That is why after we reduce all leaders of $\mathbb{C}$ from $f$ we get the polynomial depending effectively on $y_{t+1}^{(s+r_C-r_f)}$. The contradiction is the same as in [**17**, Theorem 24]. It is based on Lemma 1 and Lemma 3.

Consider a slightly another approach to proving our lemma. Let $C_i = \arg\max\limits_{C_j\in\mathbb{C}} \mathrm{ord}(C_j, y_{t+1})$. Denote by $P(s)$ the elements of $P$ of the order less than of equal to $s$. The set $P(s)$ is a prime ideal in the correspondent polynomial ring. We prove that the following system $Y'$ of differential variables is algebraically independent over $k$ modulo $P(s)$ in the field of fractions $L(s) = \mathrm{Quot}(R/P(s))$ for all sufficiently large $s\in\mathbb{N}$:

$$Y' = y_{t+1}, y'_{t+1},\dots,y_{t+1}^{(s)}, y_i, y'_i,\dots,y_i^{(r-1)}, y_{t+1}^{(s+1)},\dots,y_{t+1}^{(s+N-r)},$$

where $u_i = \theta_i y_i$ for some $\theta_i\in\Theta$ with $\mathrm{ord}\,\theta_i = r$, and $u_i$ is the leader of $C_i$, and $N = \mathrm{ord}(C_i, y_{t+1})$. Note that $\#Y' = (s+1)+N$.

These variables are, indeed, algebraically independent. If $r \geqslant N$ then $Y' = \{y_{t+1},\ y'_{t+1},\dots,\ y_{t+1}^{(s)},\ y_i, y'_i,\dots,y_i^{(r-1)}\}$ and $\#Y' = s+1+N$ with $Y' \subset L(s)$. So, assume that $r < N$. Prove that $Y'\subset L(s)$. We only need to show that $\{y_{t+1}^{(s+1)},\dots,y_{t+1}^{(s+N-r)}\}\subset L(s)$. Differentiating $C_i$ $(s-N+1)$ times we get a linear polynomial in $y_{t+1}^{(s+1)}$ with non-zero leading coefficient by Lemma 3. Then $y_{t+1}^{(s+1)}$

belongs to $L(s)$. Continuing this process we differentiate $C_i$ $(s-r)$ times and obtain that $y_{t+1}^{(s+N-r)} \in L(s)$.

Finally, it should be pointed out that $\dim P(s) = (s+1)+h$ for all sufficiently large $s$ (see [**13**] for a proof of this fact). Thus, we have $N \leqslant h$. □

Note that Lemma 4 is actually a generalization of Sadik's result. So, now we are ready to prove a final bound for a characteristic set of a radical differential ideal and to obtain an algorithm computing this set.

THEOREM 4. *Let $I$ be a radical differential ideal in $k\{y_1,\ldots,y_l\}$ with a characteristic set $\mathbb{C} = C_1,\ldots,C_p$. Let $I = \bigcap_{i=1}^n [\mathbb{C}_i] : H_{\mathbb{C}_i}^\infty$ be a characteristic decomposition w.r.t. an orderly ranking with $\mathbb{C}_i = C_i^1,\ldots,C_i^{p_i}$. Let $h$ be the maximal order of $\mathbb{C}_i$ for $1 \leqslant i \leqslant n$. Then the lowest differentially autoreduced subset of a characteristic set of the ideal*

$$I' = \bigcap_{i=1}^n (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H_{\mathbb{C}_i}^\infty$$

*is a characteristic set of $I$ w.r.t. the orderly ranking.*

PROOF. Let $I = \bigcap_{j=1}^a P_j$ be a minimal prime decomposition. Suppose that for some $i$, $1 \leqslant i \leqslant p$, we have $\operatorname{ord}(C_i, y_{j_i}) > h$, where $\theta y_{j_i} = u_i$ is the leader of $C_i$ and $\theta \in \Theta$. Note that $i \geqslant 2$, because the leader of the differential polynomial $C_1$ appears among the leaders of characteristic sets of the ideals $P_j$.

Let $I_i$ be the initial of $C_i$ and $I_i \in P_1,\ldots,P_t$ and $I_i \notin P_{t+1},\ldots,P_a$. Let us concentrate our attention at $P_{t+1},\ldots,P_a$. Denote a characteristic set of $P_j$ w.r.t. the orderly ranking by $\mathbb{B}_j$. According to Lemma 1 there exists an element with the leader $u'_j$ in each $\mathbb{B}_j$ such that $u_i = \theta_{i,j}u'_j$ for some $\theta_{i,j} \in \Theta$ and $t+1 \leqslant j \leqslant a$.

Consider a particular $P_r$ for some $r$, $t+1 \leqslant r \leqslant a$. Let $u_s = \theta_s y_s$ for all $1 \leqslant s \leqslant p$. Note that $C_i$ depends on $U = \{u_1,\ldots,u_{i-1}\}$, its leader $u_i$, and some non-leaders $V = \{v_1,\ldots,v_q\}$. Introduce a "shifted" function ord′ in such a way. Let $h' = \max\limits_{1\leqslant j\leqslant i} \operatorname{ord}(C_i, y_j)$. Put $\operatorname{ord}'\theta y_j = \operatorname{ord}\theta + h' - \operatorname{ord}(C_i, y_j)$.

Hence, the ord′-orderly ranking $>$ appears on the set $\tilde{U}$ of differential variables such that some their derivatives belong to $\Theta U$. Let $\mathbb{D} = D_1,\ldots,D_r$ be a characteristic set of $P_r$ w.r.t. the ranking $\tilde{U} >_{el} u_i >_{el} \tilde{V}$, where $\tilde{V}$ denotes the set of differential variables such that some their derivatives belong to $\Theta V$. According to Lemma 4 we have $\operatorname{ord}(D_q, y_{j_i}) \leqslant h$ for all $q$, $1 \leqslant q \leqslant r$. Moreover, $\operatorname{ord}(D_s, y_j) \leqslant \operatorname{ord}(C_i, y_j)$ for all $j$, $1 \leqslant j \leqslant j_i - 1$, and some $s$, $1 \leqslant s \leqslant r$. We have two possible cases.

The first one is that $\operatorname{ord}(D_s, y_j) = \operatorname{ord}(C_i, y_j)$ for some $1 \leqslant j \leqslant j_i - 1$. Then, either $\operatorname{ord}(C_i, y_{j_i}) \leqslant h$ and we have nothing to prove or eliminating $y_j$ one can find another $D_s$ in $P_i$ with the property $\operatorname{ord}(D_s, y_j) < \operatorname{ord}(C_i, y_j)$ for all $j$, $1 \leqslant j \leqslant j_i - 1$. That is the second case. In this one we can put

$$\widehat{C_i} = \prod_{\alpha=t+1}^a D_{s_\alpha,\alpha},$$

where $D_{s_\alpha,\alpha}$ comes from $P_\alpha$.

Finally, multiplying $\widehat{C_i}$ by $I_i$ we obtain an element of $I$ that is not reducible by $\mathbb{C}$. Contradiction. Thus, we know an upper bound for the orders of $C_i$, $1 \leqslant i \leqslant p$.

Moreover, by Theorem 3 we have $\max\limits_{1\leqslant j\leqslant a} \operatorname{ord}\mathbb{B}_j = \max\limits_{1\leqslant j\leqslant n} \operatorname{ord}\mathbb{C}_j$. By Theorem 1 and Lemma 1 we have

$$[\mathbb{C}_i] : H^\infty_{\mathbb{C}_i} \cap k[\theta_i y_i, \operatorname{ord}\theta_i y_i \leqslant h] = (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H^\infty_{\mathbb{C}_i}$$

and $\operatorname{ord} u_i \leqslant h$ for each $i$, $1 \leqslant i \leqslant n$. Thus, $I \cap k[\theta_i y_i, \operatorname{ord}\theta_i y_i \leqslant h] = I'$.

Here we used the fact that the set of leaders of any characteristic set of the ideal $P$ coincides with the one of $\mathbb{C}_i$ if $P$ is a minimal prime of $[\mathbb{C}_i] : H^\infty_{\mathbb{C}_i}$. This is true due to Theorem 2 and Theorem 3. If $P$ is a minimal prime of $I$ then $P$ is a minimal prime of some characteristic component $[\mathbb{C}_i] : H^\infty_{\mathbb{C}_i}$ of $I$ and vice versa. Hence, our problem is purely commutative algebraic now.

In order to get a characteristic set of the ideal $I$ it is sufficient to compute an algebraic characteristic set $\mathbb{C}'$ of $I'$ and then find in $\mathbb{C}'$ the lowest differentially autoreduced subset $\mathbb{C}$. This set differentially reduces the ideal $I'$ to zero. Thus, $\mathbb{C}$ is a characteristic set of $I$. □

Remark 2. The bound obtained in Theorem 4 most probably is not true for partial derivatives. At least, the method of “shifted” rankings does not work in non-ordinary cases. This will be shown in Example 7.

**3.2. Algorithm.** As an immediate consequence of Theorem 4 we have the following algorithm. Let some orderly ranking be fixed.

Algorithm 1. Ordinary Characteristic Set Computation
Input: *a finite set $F$ of ordinary differential polynomials.*
Output: *characteristic set of $\{F\}$ in Kolchin's sense.*

- *Let* $\mathfrak{C} = \chi$-Decomposition$(F)$ *and* $\mathfrak{C} = \mathbb{C}_1, \ldots, \mathbb{C}_n$ *with* $\mathbb{C}_i = C_i^1, \ldots, C_i^{p_i}$.
- *Let* $\mathbb{C}_i = C_1^i, \ldots, C_{p_i}^i$ *for each* $1 \leqslant i \leqslant n$.
- *Let* $h = \max\limits_{1\leqslant i\leqslant n} \operatorname{ord}\mathbb{C}_i$.
- *Compute* $I' = \bigcap_{i=1}^n (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H^\infty_{\mathbb{C}_i}$.
- $\mathbb{C}' :=$ *an algebraic characteristic set of the ideal* $I'$.
- *Return the differentially autoreduced subset of* $\mathbb{C}'$ *with the lowest rank.*

Remark 3. The last steps of Algorithm 1 can be performed by means of computations discussed in [**3**] and [**6, 7, 8**]. More precisely, one can compute each $I_i = (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H^\infty_{\mathbb{C}_i}$ using the Rabinovich trick and the elimination technique. Then, the intersection of the ideals $I' = \bigcap_{i=1}^n I_i$ has to be computed. The solutions to these two problems are presented in [**3**]. Finally, an algorithm for computing an algebraic characteristic set of $I'$ is given in [**6, 7, 8**].

Remark 4. Algorithm $\chi$-Decomposition used in Algorithm 1 can be replaced by Algorithm Rosenfeld_Groebner presented in [**4**] and [**5**] and implemented in Maple.

## 4. Partial differential case

**4.1. Theoretical bases.** We are *not* in the ordinary case now. Introduce a special class of radical differential ideals that are good in the computational sense.

Definition 4. We say that a radical differential ideal $I$ satisfies the *property of consistency* iff there exists a characteristic set $\mathbb{C} \subset I$ such that $1 \notin [\mathbb{C}] : H^\infty_{\mathbb{C}}$.

It is clear that any characterizable radical differential ideal satisfies the property of consistency. Moreover, it follows from Theorem 2 and Theorem 3 that any proper regular differential ideal (an ideal of the form $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ for a coherent autoreduced set $\mathbb{A}$) satisfies this property. Consider an example of non-regular radical differential ideal satisfying the property of consistency.

EXAMPLE 2. Let $\mathbb{A} = x(x-1)$, $xy$, $xz$ in $k\{x, y, z\}$ with $x < y < z$. We have $1 \notin [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. Consider the minimal prime decomposition:

$$\{\mathbb{A}\} = [x] \cap [x-1, y, z].$$

We see $\mathbb{A}$ is a characteristic set of $\{\mathbb{A}\}$. Then the radical differential ideal $\{\mathbb{A}\}$ satisfies the property of consistency. Nevertheless, since minimal primes of $\{\mathbb{A}\}$ have different sets of leader, $\{\mathbb{A}\}$ is not a regular ideal due to Theorem 2 and Theorem 3.

So, we are ready to prove Theorem 5. Let $\theta y_i$ be a differential variable in $k\{y_1, \ldots, y_l\}$. Then, by definition its order equals $\operatorname{ord}\theta$.

THEOREM 5. *Let $I$ be a radical differential ideal in $k\{y_1, \ldots, y_l\}$ satisfying the property of consistency and a characteristic set $\mathbb{C} \subset I$ with $1 \notin [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$.*

(1) *Let $U$ be the set of leaders of $\mathbb{C}$ and $U'$ be the set of leaders of any characteristic decomposition of $I$. Then $U \subset U'$.*
(2) *Let $I = \bigcap_{i=1}^{n} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ be a characteristic decomposition w.r.t. an orderly ranking with $\mathbb{C}_i = C_i^1, \ldots, C_i^{p_i}$. Let $h$ be the maximal order of differential variables appeared in the elements of $\mathbb{C}_i$ for $1 \leqslant i \leqslant n$. Then the lowest differentially autoreduced subset of a characteristic set of the ideal $\bigcap_{i=1}^{n} (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H_{\mathbb{C}_i}^{\infty}$ is a characteristic set of $I$ w.r.t. the orderly ranking.*

PROOF. We have $[\mathbb{C}] \subset I \subset [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Consider the minimal prime decomposition $I = \bigcap_{i=1}^{m} P_i$. We have

$$I = \bigcap_{i=1}^{n} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty} = \bigcap_{i=1}^{m} P_i.$$

Some components of the characteristic decomposition may appear to be unnecessary. Let $I = \bigcap_{i=1}^{k} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ be a minimal characteristic decomposition, that is, $I \neq \bigcap_{i=1, i\neq j}^{k} [\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ for all $1 \leqslant j \leqslant k$. Let $U'$ be the union of leaders of $\mathbb{C}_i$ for $1 \leqslant i \leqslant k$. If $P$ and $P_j$ are prime ideals for each $1 \leqslant j \leqslant t$ and $P \supset \bigcap_{i=1}^{t} P_i$ then $P \supset P_i$ for some $1 \leqslant j \leqslant t$ (see [**1**, Proposition 1.11]). Thus, if $P$ is a minimal prime of $I$ then $P$ is a minimal prime of $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ for some $1 \leqslant i \leqslant k$.

We obtain that the set of leaders of any characteristic set of $P$ is equal to those of $\mathbb{C}_i$ by Theorem 2 and Theorem 3 . Hence, the union of leaders of characteristic sets of minimal primes of $I$ is equal to $U'$. Include $[\mathbb{C}] : H_{\mathbb{C}}^{\infty}$ into a characteristic decomposition of $I$. For this purpose represent $\mathbb{C}$ as an output of Coherent-Autoreduced algorithm. Thus, we have $I = [\mathbb{C}] : H_{\mathbb{C}}^{\infty} \cap [\mathbb{B}_2] : H_{\mathbb{B}_2}^{\infty} \cap \ldots \cap [\mathbb{B}_r] : H_{\mathbb{B}_r}^{\infty}$. Denote the set of leaders of $\mathbb{C}$ by $U$.

Let $P$ be a minimal prime of $[\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Then $P$ is a minimal prime of $\{\mathbb{C}\}$ (see [**9**, page 644]). Thus, $P$ is a minimal prime of $I$, because $\{\mathbb{C}\} \subset I \subset [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Since $P$ is a minimal prime of $[\mathbb{C}_i] : H_{\mathbb{C}_i}^{\infty}$ for some $1 \leqslant i \leqslant k$, then the set of leaders of $\mathbb{C}_i$ is equal to $U$ and $U \subset U'$. So, we know an upper bound for the order of a characteristic set of $I$. Due to Theorem 1 and Lemma 1 we have $[\mathbb{C}_i] : H_{\mathbb{C}_i}^\infty \cap k[\theta_i y_i, \operatorname{ord}\theta_i y_i \leqslant h] = (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H_{\mathbb{C}_i}^\infty$ and $\operatorname{ord} u_i \leqslant h$ for each $1 \leqslant i \leqslant n$. Thus, we obtained the result because the end of the proof is the same as in Theorem 4. □

The main contribution of Theorem 5 is that our computations are moved into the ring of commutative polynomials in a *finite* number of variables. This is a crucial point in Algorithm 2.

Remark 5. We see that the set of leaders $U$ of $\mathbb{C}$ is not only a subset of $U'$. We have $U$ is equal to the set of leaders of some characteristic component in any characteristic decomposition of $I$. Thus, $U$ is "concentrated" in some characteristic component. We call it the "localization" property.

Remark 6. The fact that the set of leaders of $\mathbb{C}$ is equal to that of some characteristic component holds true for *any* differential ranking. This follows from the proof of Theorem 5.

Note that the condition that an ideal satisfies the property of consistency cannot be omitted in Theorem 5. To support this fact we give Example 5.

**4.2. Algorithm.** In conclusion, we obtain the following algorithm. Let some orderly ranking be fixed.

Algorithm 2. Characteristic Set Computation
Input: *a finite set $F$ of differential polynomials such that $\{F\}$ satisfies the property of consistency.*
Output: *characteristic set of $\{F\}$ in Kolchin's sense.*

- *Let* $\mathfrak{C} = \chi$-Decomposition$(F)$ *and* $\mathfrak{C} = \mathbb{C}_1, \ldots, \mathbb{C}_n$ *with* $\mathbb{C}_i = C_i^1, \ldots, C_i^{p_i}$.
- *Let* $\mathbb{C}_i = C_1^i, \ldots, C_{p_i}^i$ *for each* $1 \leqslant i \leqslant n$.
- *Let* $h = \max\limits_{1\leqslant i\leqslant n} \max\limits_{1\leqslant j\leqslant p_i} \operatorname{ord} C_i^j$.
- *Compute* $I' = \bigcap_{i=1}^n (\theta_i C_i^j, \operatorname{ord}\theta_i u_{C_i} \leqslant h) : H_{\mathbb{C}_i}^\infty$.
- $\mathbb{C}' :=$ *an algebraic characteristic set of the ideal* $I'$.
- *Return the differentially autoreduced subset of* $\mathbb{C}'$ *with the lowest rank.*

The last steps of Algorithm 2 can be performed in the same way as done in Algorithm 1.

Remark 7. Algorithm $\chi$-Decomposition used in Algorithm 2 can be replaced by Algorithm Rosenfeld_Groebner presented in [**4**] and [**5**] and implemented in Maple.

Note that $h$ from Algorithm 2 is less than $h$ from Algorithm 1 in general.

## 5. Examples

We show how to apply Algorithm 2 to a particular radical differential ideal.

Example 3. Let $I = \{(x-t)x',\ x'y',\ (x-t)(z'+y')\}$ in $\mathbb{Q}(t)\{x,y,z\}$ with $t' = 1$ and an orderly ranking $x < y < z$. We have the following decomposition:

$$I = [x-t, y'] \cap [x', z'+y'].$$

So, the maximal order of variables appeared in this decomposition is equal to 1. Hence, we need to compute the reduced Gröbner basis $G$ of the ideal $I' = (x-t, x'-1, y') \cap (x', z'+y')$. This can be done by the elimination technique:

$G$ equals the intersection of the reduced Gröbner basis w.r.t. the lexicographic ordering $x < x' < y' < z' < w$ of the ideal $(w(x-t),\ w(x'-1),\ wy',\ (1-w)x',\ (1-w)(z'+y'))$ and the ring $\mathbb{Q}(t)[x, x', y', z']$.

Finally, $G = (x-t)x',\ x'(x'-1),\ x'y',\ (x-t)(z'+y'),\ x'z'-(z'+y'),\ y'(z'+y')$. Then a characteristic set of $I'$ equals $\mathbb{C} = (x-t)x',\ (x-t)(z'+y')$ and by Theorem 5 a characteristic set of the radical differential ideal $I$ is also $\mathbb{C}$.

The following example shows the *difference* between radical differential ideals with the consistency property and an arbitrary radical differential ideal. This can be considered as a "counter-example" for Remark 5 and Remark 6.

Example 4. Let $\mathbb{A} = x(x-1),\ xy,\ (x-1)z$. We have $1 \in [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. Consider the minimal prime decomposition:

$$\{\mathbb{A}\} = [x, z] \cap [x-1, y].$$

We see $\mathbb{A}$ is a characteristic set of $\{\mathbb{A}\}$ with the set of leaders equals $U = x,\ y,\ z$. Thus, $U$ does not necessarily correspond to the *unique* characteristic component. In this example $U \subset x, z \cup x, y$ and the localization property is not valid.

Note that Theorem 5 is true for Example 4: in this case we have a restriction to the orders of the elements of a characteristic set $\mathbb{A}$. Consider a "counter-example" for Theorem 5.

Example 5. Consider a radical differential ideal defined by its characteristic decomposition:

$$I = [x-1, y] \cap [x, y^{(n)}, z^{(m)} + y]$$

in $k\{x, y, z\}$ with $x < y < z$, an orderly ranking, and $n \leqslant m$. Both of these components are prime differential ideals, because they are generated by linear differential polynomials. In addition, since they are prime, these radical differential ideals are also characterizable (see [**9**, page 646]). One can show that a characteristic set of $I$ is $\mathbb{C} = x(x-1),\ xy,\ (x-1)z^{(m+n)}$. The radical differential ideal $I$ does not satisfy the property of consistency and Theorem 5 is not true for $I$. Indeed, for $m, n > 0$ we have $m + n > \max\{m, n\}$.

So, we see that the upper bound established in Theorem 5 is *wrong* for some radical differential ideals *not satisfying* the property of consistency. Nevertheless, Example 5 is in the ordinary case and Theorem 4 can be applied. We have $\operatorname{ord}\{x, y^{(n)}, z^{(m)} + y\} = m + n$ and the maximal order of elements of $\mathbb{C}$ does not exceed $m + n$.

The following example shows that the order of a characteristic set $\mathbb{C}$ of a radical differential ideal $I$ is *not bounded* by orders of prime components of $I$.

Example 6. Consider the radical differential ideal in $k\{x, y, z_1, z_2\}$ with $x < y < z_1 < z_2$ and the orderly ranking:

$$I = [x, y] \cap [x-1, y', z_1 - y] \cap [x-2, y', z_2 - y].$$

The set $\mathbb{C} = x(x-1)(x-2),\ (x-1)(x-2)y,\ x(x-2)z_1',\ x(x-1)z_2'$ is characteristic of the ideal $I$. However, $2 = \operatorname{ord}\mathbb{C} > \max_{1 \leqslant i \leqslant 3} \operatorname{ord} P_i = 1$, where $P_i$ are minimal primes of $I$ for $1 \leqslant i \leqslant 3$.

Remark 8. We see that the bound for radical differential ideals satisfying the property of consistency is lower than one for arbitrary radical differential ideals in general. That is why radical differential ideals satisfying the property of consistency are better in the computational sense. Indeed, Algorithm 2 works faster than Algorithm 1, since we are in a lower algebraic dimension in Algorithm 2 than in Algorithm 1.

## 6. Conjecture

The following example shows that the method of "shifted" ranking used in Theorem 4 does not work in partial derivative cases.

EXAMPLE 7. Consider $k\{u\}$ with $\Delta = \{\partial/\partial x, \partial/\partial y, \partial/\partial z\}$. Let

$$P = [u_{yyyy} + u_{xxyy} + u_{xxxx} + u_z,\ u_{xxxxx} + u_{xxy} + u_{xyy} - u_{xxyyy}].$$

Consider $f = u_{xxy} + u_{xyy} + u_{xxxxx} + u_{yyyyy} + u_{xxxxy} + u_{yz} \in P$. One can prove that this polynomial cannot be an element of any characteristic set of $P$ w.r.t. "shifted" rankings. However, $f$ can be an element of a characteristic set of a radical differential ideal for which the ideal $P$ is a minimal prime component.

REMARK 9. Example 7 is not a counter-example for Theorem 4 in non-ordinary cases. It is just a counter-example for the part of the proof of the theorem.

COROLLARY 1. *One need to develope another technique to get a bound for the order of a characteristic set of a radical differential ideal in partial derivative cases and radical differential ideals not satisfying the property of consistency.*

Nevertheless, we hope that this problem can be solved by means of increasing estimates of bounds for characteristic sets of radical differential ideals. So, we *conjecture* that if $|\Delta| = m$ then the order of each element of a characteristic set of a radical differential ideal $I$ is bounded by $qh^m$, where $h = \max\limits_{1 \leqslant i \leqslant n} \operatorname{ord} \mathbb{C}_i$ and $I = \bigcap_{i=1}^n [\mathbb{C}_i] : H_{\mathbb{C}_i}^\infty$ is a characteristic decomposition w.r.t. an *orderly* ranking and $q$ is the number of different differential indeterminates $y_j$ appearing as leaders in $\mathbb{C}_i$ for $1 \leqslant i \leqslant n$.

## 7. Conclusions

We discussed an algorithm for computing a Kolchin characteristic set of an arbitrary ordinary radical differential ideal w.r.t. orderly rankings. In the partial differential case we also presented a solution to the problem of computing a characteristic set of a radical differential ideal satisfying the special property of consistency. These solutions are new and previously these problems were completely solved only in the non-differential case as to our knowledge.

The authors hope that the technique obtained in this paper can be generalized to any radical differential ideals using the ideas we presented. Another natural way of generalizing these results is to investigate non-orderly rankings such as, for example, very important elimination ones in the case of partial differential polynomials.

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