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ON CHARACTERIZABLE IDEALS
AND CHARACTERISTIC SETS

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ABSTRACT. We study the property of a radical differential ideal to be characterizable and give a criterion of characterizability. This criterion is based on the known algorithms for decomposing radical differential ideals into characterizable components. The Ritt problem and its restriction to our particular case are discussed. We establish some properties of characteristic sets of zero-dimensional radical ideals and discuss a special class of *definable* radical differential ideals. We also discuss different representations of characteristic sets of radical differential ideals.

1. INTRODUCTION

This paper is devoted to the investigation of characterizable radical differential ideals.

1.1. **Basic Definitions.** Differential algebra deals with differential algebraic equations, e.g.

$$y''^3 \sin x + y^2 = 0. \quad (1)$$

We need to find a function $y = y(x)$ satisfying equation (1). Consider a system of partial differential equations $f_1 = 0, \dots, f_m = 0$. The *ring of differential polynomials* provides an algebraic language to operate with such equations. Each differential ring R has a set of basic differentiations $\delta_1, \dots, \delta_m$ satisfying for $a, b \in R$ and $1 \leq i \leq m$ the following properties:

1. $\delta_i(a + b) = \delta_i(a) + \delta_i(b)$;
2. $\delta_i(ab) = b\delta_i(a) + a\delta_i(b)$.

The ring of differential polynomials in n variables is the differential ring of commutative polynomials $R = k[y_1, \dots, \delta_1^{k_1} \cdots \delta_m^{k_m} y_1, \dots, y_n, \dots]$, generated by y_1, \dots, y_n in this sense. The ring R as a ring of commutative polynomials is not finitely generated, but each element must depend only on a finite number

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of $\delta_1^{l_1} \cdots \delta_m^{l_m} y_i$. Note that the field k must be a differential field. For example, in equation (1) we consider $k[y, y', y'', \dots]$. The formal definition of a ring of differential polynomials is given in [10, 12].

The notion of reduction appears in constructive differential algebra. Using this notion one can check whether an element f is a consequence of the system $f_1 = 0, \dots, f_m = 0$. Using the reduction one can answer the question: Has this system a solution in some extension of k ? So, differential algebra is very useful for studying the properties of partial differential equations.

1.2. The Goal. A factorization free algorithm with a clear solution of the membership problem to radical differential ideals appeared for the first time in the paper by Boulier, Lazard, Ollivier and Petitot, [3]. Any radical differential ideal can be decomposed into characterizable components. The factorization free decomposition algorithm is discussed by Hubert [8]. The definition of a characterizable differential ideal has also been introduced in [8]. Is it possible to have only one component in the intersection (is the decomposition trivial)? This property implies that the ideal is characterizable.

The decomposition of an ideal helps us to solve many problems concerning the system of differential equations associated with the ideal and gives us the answer whether a fixed differential polynomial belongs to the ideal. The property of an ideal to be characterizable is its pure differentially algebraic characteristic. Our aim is to investigate the connection of the characterizability of ideals with the problem of triviality of the characteristic decomposition (Theorem 9). This problem is closely related to the inclusion problem. We formulate it in Section 2.2.2.

We also consider some aspects of constructing characteristic sets of ideals (Section 2.3). We investigate principal ideals concerning to this aspect. Then we discuss differential algebraic properties of radical differential ideals in Section 2.3.2. Theorem 31 describes the *representation of a special form* of a characteristic set of a radical differential ideal.

1.3. Historical aspects. This work uses results concerning the interpretations of differential algebraic properties in the language of commutative algebra. The author was inspired by the works of Hubert [8] and Kandri Rodi [4]. The work of Hubert gave the source of the investigation and both of these articles proposed interesting techniques. The problem of characterizability as far as I know, is not completely solved yet. In our paper the problem of characterizability is reformulated in the algebraic language and we discuss the problems which occur in the connection with it, e.g. we consider the construction of a characteristic set of a radical differential ideal and extend the class of characterizable ideals to *definable ideals* (we introduce and investigate this notion in Section 3). This new class can be described algorithmically.

2. CHARACTERIZABILITY CRITERION FOR A RADICAL DIFFERENTIAL IDEAL

In this section we prove a characterizability criterion in a ring of differential polynomials. The technique described in [4] and [8] helps us to cover some properties of these ideals. Then we investigate characteristic sets of radical differential ideals.

We use the following notation. For autoreduced sets we use capital letters $\mathbb{A}, \mathbb{B}, \mathbb{C}, \dots$. If we want to enumerate the elements of \mathbb{A} we write the following: $\mathbb{A} = A_1, A_2, \dots, A_m$. By f, g, h, \dots we denote polynomials. For ideals we use the following notation: I, J, P, Q . We denote by k the basic differential field. We denote by $k\{y_1, \dots, y_l\}$ the ring of differential polynomials in l variables. Let $F \subseteq k\{y_1, \dots, y_l\}$ be a finite set of differential polynomials. For the differential and radical differential ideal generated by F in $k\{y_1, \dots, y_l\}$, we use the notation $[F]$ and $\{F\}$, respectively.

We need the notion of reduction for algorithmic computations. First, we introduce a *differential ranking* on the set of differential variables of $k\{y_1, \dots, y_l\}$. This is a total ordering on the set of all θy_i with $\theta \in \Theta$ and $1 \leq i \leq l$ (here Θ is the set of all $\delta_1^{k_1} \delta_2^{k_2} \dots \delta_n^{k_n}, k_i \geq 0$) satisfying the following conditions

$$\theta u \geq u, \quad u \geq v \implies \theta u \geq \theta v.$$

In later discussions we suppose that some ranking is fixed.

We call the highest variable appearing in a differential polynomial $f \in k\{y_1, \dots, y_l\} \setminus k$ the *leader* of f ; we denote this leader by u_f and represent f as a univariate polynomial in u_f :

$$f = I_f u_f^n + a_1 u_f^{n-1} + \dots + a_n.$$

The polynomial I_f is called the *initial* of f .

Apply any $\delta \in \Theta$ to f :

$$\delta f = \frac{\partial f}{\partial u_f} \delta u_f + \delta I_f u_f^n + \delta a_1 u_f^{n-1} + \dots + \delta a_n.$$

The leading variable of δf is δu_f and the initial of δf is called the *separant* of f . We denote this separant by S_f . Note that for all $\theta \in \Theta, \theta \neq 1$, each θf has the initial equal to S_f .

We define the rank on differential polynomials. We say that $f \geq g$ iff $u_f > u_g$ or in case of $u_f = u_g$ we have $\deg_{u_g} f \geq \deg_{u_g} g$.

We say that a differential polynomial f is *partially reduced* w.r.t. g iff no proper derivative of u_g appears in f . A differential polynomial f is *reduced* w.r.t. g iff f is partially reduced w.r.t. g and $\deg_{u_g} f < \deg_{u_g} g$. Consider any subset $\mathbb{A} \subseteq k\{y_1, \dots, y_l\}$. We say that \mathbb{A} is autoreduced iff $\mathbb{A} \cap k = \emptyset$ and each element of \mathbb{A} is reduced w.r.t. all the others. Every autoreduced set is finite.

We denote by $I_{\mathbb{A}}$ and $S_{\mathbb{A}}$ the product of the initials and the separants of \mathbb{A} respectively. Denote $I_{\mathbb{A}} \cdot S_{\mathbb{A}}$ by $H_{\mathbb{A}}$. Let S be a finite set of differential

polynomials. Denote by S^∞ the multiplicative set generated by S . Let I be an ideal in R . Then $I : S^\infty$ is the set of all $a \in R$ such that $\exists s \in S^\infty : sa \in I$. If I is a differential ideal then $I : S^\infty$ is also a differential ideal.

Let $\mathbb{A} = A_1, \dots, A_r$ and $\mathbb{B} = B_1, \dots, B_s$ be autoreduced sets. Let \mathbb{A} and \mathbb{B} be arranged in order of increasing rank. We say that \mathbb{A} has lower rank than \mathbb{B} iff

- either there exists $k \leq r, s$ such that $\text{rank} A_i = \text{rank} B_i$ for $1 \leq i < k$ and $\text{rank} A_k < \text{rank} B_k$,
- or $r > s$ and $\text{rank} A_i = \text{rank} B_i$ for $1 \leq i \leq s$.

We say that $\text{rank} \mathbb{A} = \text{rank} \mathbb{B}$ iff $r = s$ and $\text{rank} A_i = \text{rank} B_i$ for $1 \leq i \leq r$.

Consider differential polynomials f and g in $R = k\{Y\}$. Let I be the differential ideal in R generated by g . Applying a finite number of differentiations and pseudo divisions one can compute the *differential partial remainder* f_1 and *differential remainder* f_2 of f w.r.t. g such that there exist $s \in S_g^\infty$ and $h \in H_g^\infty$ satisfying $sf \equiv f_1$ and $hf \equiv f_2 \pmod{I}$ where f_1 and f_2 are partially reduced and reduced respectively.

2.1. General Approach. Consider the polynomial ring $k[x_1, \dots, x_n]$ over a differential field k of characteristic zero, and an autoreduced set \mathbb{A} in this ring. Let $U, Y \subseteq \{x_1, \dots, x_n\}$ be the sets of “leaders” and “non-leaders”, respectively, of the elements of the autoreduced set $\mathbb{A} \subseteq k\{y_1, \dots, y_l\}$ (see [8] for details). Thus we can denote $k[x_1, \dots, x_n]$ by $k[Y][U]$.

Example 1. Let $\mathbb{A} = A_1, A_2 \subseteq k\{v, u_1, u_2\}$, where $A_1 = vu_1^2 + u_1 + v^2$, $A_2 = u_1u_2^3 + v$ and $v < u_1 < u_2$. We have $U = u_1, u_2$ and $Y = v$.

Let $\mathbb{A} = A_1, \dots, A_m$ be an autoreduced set. As stated above, we denote the leader of A_i by u_{A_i} for each $1 \leq i \leq m$. We want to emphasize that our notation differs from the notation used in [4].

Definition 2. [4, Definition 1.2.1] Let \mathbb{A} be an autoreduced set in $k[Y][U] = R$ and f a polynomial in $k[Y][U]$. We say that f is *invertible* w.r.t. \mathbb{A} iff f is invertible modulo (\mathbb{A}) in $k(Y)[U]$, i.e. there exist $g \in k[Y][U]$ and $h \neq 0$ in $k[Y]$ such that $f \cdot g \equiv h \pmod{(\mathbb{A})}$.

This property is very important for the “characteristic” polynomials of the autoreduced set, namely, the initials and the separants. According to [4], we say that the initials and the separants of an autoreduced set $\mathbb{A} = A_1, \dots, A_p$ are invertible iff, for each j , the initial I_j and the separant S_j are invertible modulo (A_1, \dots, A_{j-1}) and (A_1, \dots, A_j) , respectively.

The notion of a characteristic set in Kolchin’s sense in characteristic zero is important in our discussions.

Definition 3. [10, page 82] An autoreduced set with the lowest rank in an ideal I is called a *characteristic set* of I .

As it is mentioned in [10], \mathbb{A} is a characteristic set of a differential ideal I iff each element of I reduces to zero w.r.t. \mathbb{A} . Consider the definition of a characterizable radical differential ideal.

Definition 4. [8, Definition 2.6] A radical differential ideal I in $k\{y_1, \dots, y_l\}$ is said to be *characterizable* iff there exists a characteristic set \mathbb{A} of I in Kolchin's sense such that $I = [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$.

The following definition builds a bridge between differential and commutative algebra. Let v be a derivative in $k\{y_1, \dots, y_l\}$, and let \mathbb{A}_v be the set of those elements of \mathbb{A} and their derivatives that have a leader ranking strictly lower than v .

Definition 5. \mathbb{A} is *coherent* iff whenever $A, B \in \mathbb{A}$ are such that u_A and u_B have a common derivative $v = \psi u_A = \phi u_B$, then $S_B \psi A - S_A \phi B \in (\mathbb{A}_v) : H_{\mathbb{A}}^{\infty}$.

First, we formulate the following assertions.

Theorem 6. [4, Theorem 2.1.1] Let $\mathbb{A} = A_1, \dots, A_p$ be a coherent autoreduced set. Then \mathbb{A} is a characteristic set of $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ iff the initials and separants of \mathbb{A} are invertible.

Theorem 7. [4, Theorem 2.1.2] Let F be a set of differential polynomials and $\mathbb{A} = A_1, \dots, A_p$ be a coherent autoreduced set with invertible initials and separants and reduce the elements of F to zero. Then

$$\{F\} = [\mathbb{A}] : H_{\mathbb{A}}^{\infty} \cap \{F, S_1\} \cap \dots \cap \{F, S_p\}.$$

The following Proposition 8 states an important property of characteristic sets with invertible initials and separants.

Proposition 8. Let the autoreduced sets \mathbb{A} and \mathbb{C} be characteristic sets of the ideals $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ and $[\mathbb{C}] : H_{\mathbb{C}}^{\infty}$ respectively. Suppose that $\mathbb{C} \subseteq [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ and $\mathbb{A} \subseteq [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. Then \mathbb{A} and \mathbb{C} , being considered as autoreduced sets, have the same rank.

Proof. Consider the differential ideal $I = [\mathbb{A}] : H_{\mathbb{A}}^{\infty} \cap [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. We have $\mathbb{A} \subseteq I$ and $\mathbb{C} \subseteq I$. Both of these autoreduced sets reduce each element of I to zero. We obtain that \mathbb{A} and \mathbb{C} are characteristic sets of I in Kolchin's sense. Hence, they have the same rank. \square

The following Theorem 9 gives us a *characterizability criterion*. It deals with a coherent autoreduced set \mathbb{A} with invertible initials and separants in a radical differential ideal I such that $[\mathbb{A}] \subseteq I \subseteq [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. This is the main subject of the discussion in the next section. We discuss the existence of such an \mathbb{A} in I . If it does not exist, I is not characterizable. We propose the following criterion of characterizability for radical differential ideals.

Theorem 9. *Let $F \subseteq k\{y_1, \dots, y_l\}$ be a finite set of differential polynomials and $\mathbb{A} = A_1, \dots, A_p$ be a coherent autoreduced set with invertible initials and separants such that $[\mathbb{A}] \subseteq \{F\} \subseteq [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. Then $\{F\}$ is characterizable iff $[\mathbb{A}] : H_{\mathbb{A}}^{\infty} \subseteq \{F, S_i\}$, $1 \leq i \leq p$, where S_i are the separants of \mathbb{A} .*

Proof. By Theorem 6, \mathbb{A} is a characteristic set of $[\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. And, by Theorem 7, we can represent $\{F\}$ as an intersection of radical differential ideals:

$$\{F\} = [\mathbb{A}] : H_{\mathbb{A}}^{\infty} \cap \{F, S_1\} \cap \dots \cap \{F, S_p\}.$$

If $[\mathbb{A}] : H_{\mathbb{A}}^{\infty} \subseteq \{F, S_i\}$, $1 \leq i \leq p$, then $\{F\} = [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$ and $\{F\}$ is characterizable by the characteristic set \mathbb{A} .

If $[\mathbb{A}] : H_{\mathbb{A}}^{\infty} \not\subseteq \{F, S_i\}$ for some i , then $\{F\} \subsetneq [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. Then the fact that the radical differential ideal $\{F\}$ is not characterizable is a consequence of the definition of characteristic and autoreduced sets and the following inclusions (suppose that $\{F\}$ is characterizable and \mathbb{C} is its characteristic set):

$$[\mathbb{A}] \subseteq \{F\} = [\mathbb{C}] : H_{\mathbb{C}}^{\infty} \subsetneq [\mathbb{A}] : H_{\mathbb{A}}^{\infty}.$$

Both \mathbb{C} and \mathbb{A} are characteristic sets in the corresponding ideals. We have the following inclusions:

$$\mathbb{A} \subseteq [\mathbb{C}] : H_{\mathbb{C}}^{\infty} \text{ and } \mathbb{C} \subseteq [\mathbb{A}] : H_{\mathbb{A}}^{\infty}.$$

Let $f \in [\mathbb{A}] : H_{\mathbb{A}}^{\infty} \setminus [\mathbb{C}] : H_{\mathbb{C}}^{\infty}$. By Proposition 8 both \mathbb{A} and \mathbb{C} have the same set of leaders and the degrees of the corresponding leaders are equal. We have a one-to-one correspondence between the first steps of the reduction of f by \mathbb{A} and \mathbb{C} , but such an element f cannot exist. Indeed, let $g \neq 0$ be a reduction of f w.r.t. \mathbb{C} . We have $g \in [\mathbb{A}] : H_{\mathbb{A}}^{\infty}$. Then g is reducible to zero w.r.t. \mathbb{A} . So, we can apply at least one step of reduction to g by \mathbb{A} . Thus, g is reducible w.r.t. \mathbb{C} , contradiction. Hence, $\{F\}$ is not characterizable. \square

Consider some examples, which say us that to find out whether an ideal is characterizable, is not a trivial task. Let us illustrate this by examples of principal ideals.

Example 10. Let $F = y'^2 - y^2$ be a polynomial in $k\{y\}$. Then we have the decomposition

$$\{y'^2 - y^2\} = [y'^2 - y^2] : (y')^{\infty} \cap \{y'^2 - y^2, 2y'\}.$$

Since $\{y'^2 - y^2, 2y'\} = [y]$ and $[y'^2 - y^2] : (y')^{\infty} \subseteq [y]$ (this fact can be checked directly) we have $\{y'^2 - y^2\} = [y'^2 - y^2] : (y')^{\infty}$. Hence, $\{y'^2 - y^2\}$ is a characterizable ideal. This is an example of a non-prime characterizable radical differential ideal.

Example 11. Let $F = y'^2 + y$ be a polynomial in $k\{y\}$. Then we have the decomposition

$$\{y'^2 + y\} = [y'^2 + y] : (y')^{\infty} \cap \{y'^2 + y, 2y'\}.$$

Again we have $\{y'^2 + y, 2y'\} = [y]$. Nevertheless, $\{y'^2 + y\}$ is not characterizable since $2y'' + 1 \in [y'^2 + y] : (y')^\infty$, $2y'' + 1 \notin [y]$ and $[y'^2 + y] : (y')^\infty \not\subseteq [y]$.

Remark 12. This is an example of a non-characterizable radical differential ideal, generated by an irreducible polynomial. Hence, $\{\text{irreducible polynomial}\}$ is not necessary a prime differential ideal.

Remark 13. The property of a radical ideal $(\mathbb{A}) : H_{\mathbb{A}}^\infty$ to be characterizable in $k[Y][U]$ does not imply the property of the radical differential ideal $[\mathbb{A}] : H_{\mathbb{A}}^\infty$ to be characterizable in $k\{y_1, \dots, y_l\}$. This is a consequence of Example 11.

2.2. Algorithmic problems. First, consider the following problem. Let I be a radical differential ideal generated by a finite set of differential polynomials F , i.e. $I = \{F\}$. How to answer a very natural question, whether I is a characterizable ideal? Let us execute the following steps. Find a coherent autoreduced set \mathbb{A} such that $[\mathbb{A}] \subseteq \{F\} \subseteq [\mathbb{A}] : H_{\mathbb{A}}^\infty$. This set can be obtained by the well-known algorithm originating in the Gröbner bases technique, namely Coherent-Autoreduced [8, Algorithm 5.1]. If the initials and separants of \mathbb{A} are invertible, we can apply Theorem 9. Note that one can algorithmically check the invertibility of separants and initials. For this purpose we can use [4, Algorithm **Invert**]. Then we only need to check the inclusions described in Theorem 9.

This leaves the following problem: How to deal with non-invertible separants and initials? Our aim is to apply Theorem 9, that is why we need to obtain a coherent autoreduced set with invertible initials and separants. In some cases of non-characterizable ideals it is impossible to get such a set. We discuss this problem in Section 3. Nevertheless, this problem does not appear in the case of principal ideals as is proved in the following discussions.

2.2.1. Essential properties of principal ideals. We show that the problem of non-invertible separants can be avoided. Note that, according to Definition 2, it is impossible to consider the property of invertibility of the initial in a principal ideal.

Proposition 14. *Let \mathbb{A} consist of one differential polynomial A . Then the separant of A is invertible iff A is a square-free polynomial in $k(Y)[U]$.*

Proof. We have to consider the ring $k(Y)[U]$ where U has only one element, namely the leader of A . Denote by A' the derivative of A . From Definition 2 we obtain the following statement: The separant of a differential polynomial A is invertible iff there exist polynomials $B, C \in k[Y][U]$ and $D \in k[Y] \setminus \{0\}$, with D depending only on non-leaders, such that

$$BA' = D + CA.$$

Hence, we have obtained the required assertion. \square

Corollary 15. *Let F be a differential polynomial in $k\{y_1, \dots, y_l\}$. Then there exists a differential polynomial A such that $[A] \subseteq \{F\} \subseteq [A] : H_A^\infty$ and the separant of A is invertible. Moreover, this polynomial can be obtained in a finite number of steps.*

Proof. Let $F = p_1^{a_1} \cdot \dots \cdot p_m^{a_m}$, where p_i are irreducible polynomials in $k[Y][U]$. Since the ideal $\{F\}$ is radical, we can apply the square-free procedure to F and obtain a differential polynomial $A = p_1 \cdot \dots \cdot p_m$. Hence, according to Proposition 14, we obtain the coherent autoreduced set $\mathbb{A} = A$ with invertible separants. \square

Corollary 16. *In the case of principal ideals, we reduce the problem of determining the characterizability of an ideal to the inclusion problem.*

Proof. Using Corollary 15, we can apply Theorem 9. Hence, we should only investigate the inclusion problem. \square

Proposition 17. *If $A = I_A u$, where u is the leader and I_A is the initial of \mathbb{A} , respectively, then we can exclude the initial from the product in the fractional ideal. In fact, $[A] : H_A^\infty = [u] : 1^\infty$. This is a consequence of the definition of fractional ideal. In this case the ideal is characterizable iff $u \in \{I_A\}$. This consideration will help us in the future.*

Proof. Indeed, $\mathbb{A} = A$ is a coherent autoreduced set with invertible separants. Hence, \mathbb{A} is a characteristic set of $[\mathbb{A}] : H_{\mathbb{A}}^\infty$. We can apply Theorem 9. $\{\mathbb{A}\} \subseteq [\mathbb{A}] : H_{\mathbb{A}}^\infty \cap \{A, S_A\} = [u] \cap \{I_A u, I_A\} = [u] \cap \{I_A\}$. Thus, we have proved the proposition. \square

2.2.2. The inclusion problem. We discuss the inclusion problem in the case of principal ideals. This problem is also studied in [10, 11, 12]. In our particular task we have a restriction of this problem. We need to determine the minimal decomposition of $\{f\}$ into regular components. This problem can be solved by means of minimal regular decomposition.

In [9] a minimal regular decomposition algorithm is presented ([9, Algorithm 7.3]). The solution is based on the preparation polynomial technique and the Low Power Theorem for regular differential polynomials ([9, Proposition 5.2, Theorems 6.1, 6.2]).

To apply the criterion obtained in this paper (Theorem 9) we only need to execute the minimal decomposition algorithm and exclude unnecessary components of the decomposition of $\{f\}$. According to Theorem 9 the radical differential ideal $\{f\}$ is characterizable iff we have only one component of its minimal regular decomposition obtained by [9, Algorithm 7.3], because the first component returned by this algorithm is a characteristic set of $\{f\}$ with invertible separant.

Remark 18. We have really discussed the solution of the problem of characterizability, when we can determine for a radical differential ideal I some square-free differential polynomial f such that $[f] \subseteq I \subseteq [f] : S_f^\infty$.

Let us return to the general situation. To solve the inclusion problem means to determine for any coherent autoreduced set \mathbb{A} and a radical differential ideal $I = \{F\}$ in $k\{y_1, \dots, y_l\}$ whether $[\mathbb{A}] : H_{\mathbb{A}}^\infty \subseteq \{F\}$ or not. If we know generators of $[\mathbb{A}] : H_{\mathbb{A}}^\infty$ as a radical differential ideal, this problem can easily be solved by means of the characteristic decomposition of $\{F\}$. From now on an ideal is not necessarily principal.

As we have already seen, checking the inclusion is very difficult even in the case of principal ideals. Thus, we shall make our assertions modulo the inclusion problem. This problem is the restriction of the well-known Ritt problem. If we know the generators of the ideal, we can easily check the inclusion of it. “Easily” means that there exists an algorithm. If we want to check $I_1 \subseteq I_2$ and generators of I_1 are known, we just check that each generator belongs to I_2 using, e.g., the characteristic decomposition of I_2 . In our particular case we have $I_2 = \{F\}$ and its characteristic decomposition is given. Hence, under the above assumption, in the decomposition of the radical differential ideal in Theorem 9, one can exclude unnecessary ideals from the set of $\{F, S_i\}$.

2.2.3. Corollary. According to the previous discussions we need to find the characteristic set of an ideal. In the next section we divide this problem into two cases. The first case is the case of definable ideals and the second one is the opposite case.

2.3. Some aspects of constructing characteristic sets. In this section we study the properties of characteristic sets of radical differential ideals. In particular, we investigate the structure of characteristic sets in the case of one “non-invertible” initial (Corollary 29 and Theorem 31). We also discuss an approach to a computation of a characteristic set of a radical differential ideal.

2.3.1. General discussions. Our aim is to present an algorithm which gives us a characteristic set in some cases. According to the previous discussions this task can be solved directly for each principal ideal applying the square-free procedure. The following Theorem 19 states that this technique can be applied to the general situation.

Theorem 19. *Let I be a radical zero-dimensional ideal in $k[u_1, \dots, u_n] = R$ with a reduced Gröbner basis $\mathbb{A} = f_1, \dots, f_n$ w.r.t. the lex-ordering $u_1 < \dots < u_n$. Let $1 \notin I$ and u_i be the leader of f_i , $1 \leq i \leq n$. Then S_{f_i} is invertible.*

Proof. We have that \mathbb{A} is a Gröbner basis of I w.r.t. $u_1 < \dots < u_n$. For an i , $1 \leq i \leq n$, consider the polynomial f_i and the lex-ordering where u_i is the minimal element. Let $\mathbb{B} = g_1, \dots, g_m$ be the reduced Gröbner basis of I w.r.t.

this ordering. The polynomial g_1 depends only on u_i and has the lowest degree in u_i among the elements of $I \cap k[u_i]$.

Note that g_1 is a square-free polynomial. Indeed, suppose that $g_1 = h^2 \cdot f$. Then $h \cdot f \in I$, but $\deg_{u_i} h \cdot f < \deg_{u_i} g_1$. Hence, $h \cdot f$ is not reducible to zero w.r.t. \mathbb{A} . Thus, $1 \in (g_1, S_{g_1})$ and $1 = ag_1 + bS_{g_1}$ for some polynomials a and b . Let $g_1 = a_1f_1 + \dots + a_if_i$. Then $S_{g_1} = (a_1f_1 + \dots + a_if_i)_{u_i}$. We obtain that $S_{g_1} \in (f_1, \dots, f_i, S_{f_i})$. Finally, $1 \in (f_1, \dots, f_i, S_{f_i})$. Since $1 \notin I$, this condition implies that S_{f_i} is invertible. \square

The following assertion obtained in [4] is very useful for dealing with the invertibility of polynomials.

Proposition 20. [4, Corollary 1.2.1] *Let $\mathbb{A} = A_1, \dots, A_p$ be an autoreduced set such that the initials are invertible and let f be in $k[Y][U]$. Then f is not invertible w.r.t. \mathbb{A} iff there is g in $k[Y][U]$ such that $f \cdot g \in (\mathbb{A})$ and g is not reducible to zero w.r.t. \mathbb{A} .*

Remark 21. Since \mathbb{A} with invertible initials is a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^\infty$, Proposition 20 means that the non-invertible elements and the zero divisors of $k(Y)[U]/(\mathbb{A})$ are the same sets.

The main strategy is to obtain, at each step of the algorithm, a set whose rank is lower than the rank of the set in the previous step. Finding this procedure, we present a characteristic set of an ideal. Indeed, we can use the criterion described in [4] for the coherent autoreduced set to be a characteristic set: the initials and separants should be invertible (Theorem 6). This technique works correctly only in the case of *definable* ideals. We describe this notion in Section 3. So, we should present a strictly decreasing sequence of subsets of the source radical differential ideal. Nevertheless, we use another technique in our attempts to obtain a characteristic set of an ideal in the general case. Theorem 19 shows that we should not consider invertibility of separants dealing with zero-dimensional radical ideals. In our computations we use the notion of *median* introduced in [1].

Let I be an ideal in $k[x_1, \dots, x_n]$. The median of I is an algebraic characteristic set of I . Equivalent definitions of the median are discussed in [1]. Note that, since a median is initially reduced (I_{A_k} is reduced w.r.t. A_j for $j < k$), we can reduce other coefficients of the elements of the median and obtain an autoreduced median. We introduce the notion of *autoreduced radical median*. This is an autoreduced median of $\sqrt{(\mathbb{A})}$. Consider an autoreduced set \mathbb{A} in $k\{y_1, \dots, y_l\}$ and the corresponding polynomial ring $k[Y][U]$. The lex-ordering from the corresponding ring of differential polynomials induces also an ordering on the set of monomials. First, we compute a median of the ideal $(\mathbb{A}) \subseteq k[Y][U]$. Then we apply the autoreduction described in the previous paragraph and obtain an *autoreduced median* \mathbb{A}' in $k\{y_1, \dots, y_l\}$. Note that $\mathbb{A}' \leq \mathbb{A}$.

Remark 22. According to Theorem 19, we may suppose that the autoreduced radical median of a zero-dimensional ideal has *invertible* separants. Since we check the invertibility in $k(Y)[U]/I$ we apply this theorem when the non-leaders are in the field of quotients.

The following Proposition 26 shows us that if we consider radical autoreduced medians we should only investigate the invertibility of initials. First, we need to formulate the following facts.

Theorem 23. [8, Theorem 3.2] *Let \mathbb{A} be an autoreduced set of $k[Y][U]$. If $1 \notin (\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ then any minimal prime of $(\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ admits the set Y of non-leaders of \mathbb{A} as a transcendence basis. More specially, any characteristic set of a minimal prime of $(\mathbb{A}) : S_{\mathbb{A}}^{\infty}$ has the same set of leaders as \mathbb{A} .*

Proposition 24. [8, Proposition 3.3] *Let \mathbb{A} be an autoreduced set. Then $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$ is radical if and only if $(\mathbb{A}) : I_{\mathbb{A}}^{\infty} = (\mathbb{A}) : H_{\mathbb{A}}^{\infty}$.*

Theorem 25. [4, Theorem 1.2.2] *Let $\mathbb{A} = A_1, \dots, A_p$ be an autoreduced set. Then \mathbb{A} is a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$ in $k[Y][U]$ iff the initials of \mathbb{A} are invertible.*

Proposition 26. *Let I be a radical ideal in $k(Y)[U]$ with characteristic set $\mathbb{A} = A_1, \dots, A_p$. Let $1 \notin I$ and the initials of \mathbb{A} be invertible. Then the separants of \mathbb{A} are also invertible.*

Proof. Since the initials of \mathbb{A} are invertible, $(\mathbb{A}) = (\mathbb{A}) : I_{\mathbb{A}}^{\infty}$ in $k(Y)[U]$. We have $(\mathbb{A}) : I_{\mathbb{A}}^{\infty} = (\mathbb{A}) \subseteq I$. By Theorem 25, \mathbb{A} is a characteristic set of $(\mathbb{A}) : I_{\mathbb{A}}^{\infty}$. By the conditions, \mathbb{A} reduces the elements of I to zero. Thus, $I = (\mathbb{A}) : I_{\mathbb{A}}^{\infty}$.

Since I is a radical ideal, we have by Proposition 24, $(\mathbb{A}) : I_{\mathbb{A}}^{\infty} = (\mathbb{A}) : H_{\mathbb{A}}^{\infty}$. According to Theorem 23, $\dim(\mathbb{A}) : H_{\mathbb{A}}^{\infty} = 0$ in $k(Y)[U]$. Thus, (\mathbb{A}) is a zero-dimensional radical differential ideal in $k(Y)[U]$. By Theorem 19, the separants of \mathbb{A} are invertible. \square

2.3.2. Differential algebraic properties of reducible elements. To obtain a characteristic set of a radical differential ideal we should investigate the properties of elements reducible to zero w.r.t. autoreduced sets. Note that the result of the reduction w.r.t. a characteristic set of an ideal is uniquely determined. Hence, we may choose a specific algorithm of a normal form. Then we should obtain an autoreduced set, which reduces, using this fixed algorithm of reduction, each element of the ideal to zero.

We choose the algorithm of reduction described in [12]. According to it we should first reduce the polynomial by the higher element of the autoreduced set and multiply the polynomial that we want to reduce by the lowest power of the initial or the separant. If we do not fix an algorithm of reduction we obtain Example 27. We describe the class of reduction considered in the following example. Let us reduce a differential polynomial g by an autoreduced set $\mathbb{A} =$

A_1, \dots, A_p . At each step of reduction the leading term of the corresponding derivative of A_i should appear in g' , where g' is the result of the latest step of our algorithm.

As the following example shows, the set of elements reducible to zero w.r.t. an autoreduced set is not stable to differentiations in general. More precisely, there exists an algorithm satisfying the conditions described in the previous paragraph which reduces some differential polynomial f to zero, and there is no such an algorithm which reduces f_x to zero.

Example 27. Let $\mathbb{A} = \{u_{xxx}^2 + u, u_{xy}^2 + 2u_x u_{xy}\}$ and $u_{xy} < u_{xxx}$. Consider the element f of the ideal generated by \mathbb{A} given by $f = (u_y + u)(u_{xxx}^2 + u)_y - u_{xxx}(u_{xy}^2 + 2u_x u_{xy})_{xx}$. This polynomial is reducible to zero, since we present the way how to reduce it. However, the polynomial f_x is not reducible to zero by Ritt's algorithm. The main reason is that \mathbb{A} is not a coherent set.

Remark 28. Example 27 shows us that if \mathbb{A} is an autoreduced set we cannot assert that the sum of two elements, which are both reducible to zero polynomials, is also reducible to zero. Indeed, both $((u_y + u)(u_{xxx}^2 + u)_y)_x$ and $(-u_{xxx}(u_{xy}^2 + 2u_x u_{xy})_{xx})_x$ are reducible to zero w.r.t. \mathbb{A} .

In Section 2.2.1 the problem of construction of a characteristic set for principal ideals has been *completely solved*. We discuss some aspects of this problem in the case of radical differential ideals that have an autoreduced set containing only two differential polynomials and reducing the generators of the ideal to zero. First, we describe a procedure that computes an autoreduced set with desirable properties. Suppose we have a radical differential ideal $I = \{f_1, \dots, f_n\}$.

Let $\mathbb{B} = \mathbb{A} = f_1, \dots, f_n$. We shall apply step by step the following instructions to the current set \mathbb{A} and put the obtained result to \mathbb{A} again until we have nothing to change in \mathbb{A} . As we shall see we either stop or present an autoreduced set with a lower rank. Since the order on autoreduced sets is artinian, we stop at a finite set. In our computations we shall use an extended coherent-autoreduced procedure.

Choosing an autoreduced set with the lowest rank we shall try to add polynomials with invertible initials, i.e. if we have two polynomials of the same rank and the first (for example) of them has an invertible initial, we add just the first one and invert its initial. If all of the polynomials to add have non-invertible initials, we add the polynomial with the lowest rank of the initial. So, if we have \mathbb{A} and \mathbb{B} we should first compute the reduced Gröbner basis of $\sqrt{(\mathbb{B})}$ (see [2] or [7] for the algorithms for these computations). Then we apply the *extended coherent-autoreduced procedure* and add all additional polynomials to \mathbb{A} . We put the result into \mathbb{A} if the new \mathbb{A} has a lower rank than the previous one.

Let us discuss invertibility of the initials and separants in \mathbb{A} . Note that if I_{A_2} is invertible, we may assume that it depends only on non-leaders. If the initial depends only on non-leaders there are no problems in reduction using A_2 . To investigate the second case we suppose that I_{A_2} is not invertible. So, $I_{A_2}h = A_1g$ for some h and g , where h is a reduced polynomial. Multiplying this equality by the corresponding polynomial, we may assume that $A_1 = I_{A_2}h_1$ in $k(Y)[U]$. Consider the polynomial A_2 . If some coefficient of it, considered as a univariate polynomial of u_{A_2} , is invertible, we can reduce the rank of \mathbb{A} multiplying by the corresponding polynomial, which moves to zero I_{A_2} modulo (A_1) . So, each coefficient is non-invertible.

Consider the prime decomposition $A_1 = p_1 \cdot \dots \cdot p_m$. Since I_{A_2} is not invertible, $I_{A_2} = p_1 \cdot \dots \cdot p_l$ (for some numeration of $\{p_i\}$). Suppose that some coefficient a_j of A_2 is not divisible by I_{A_2} , e.g. p_1 does not divide a_j . Then $p_{l+1} \cdot \dots \cdot p_m a_j \notin (A_1)$, but $p_{l+1} \cdot \dots \cdot p_m I_{A_2} \in (A_1)$. So we can either reduce the rank of \mathbb{A} or each coefficient of A_2 is divisible by I_{A_2} in $k(Y)[U]$.

Corollary 29. *In our calculations we may assume that $\mathbb{A} = I_{A_2}h_1, I_{A_2}h_2$.*

2.4. Representation of a characteristic set. In this section we generalize Corollary 29. In order to do this, consider the notion of a *differential chain*.

Definition 30. [5, Definition 3.2] A finite subset \mathbb{A} of $k\{y_1, \dots, y_l\}$ is said to be a differential chain iff:

1. No element of \mathbb{A} belongs to k .
2. Each element of \mathbb{A} is partially reduced w.r.t. all the others.
3. The leaders of the elements of \mathbb{A} are pairwise distinct.
4. No initial of an element of \mathbb{A} is reduced to zero by \mathbb{A} .

It should be noticed that every autoreduced set is a differential chain. Let $\mathbb{A} = A_1, \dots, A_i$ be a differential chain. Suppose that $i \geq 2$ and the differential chain A_1, \dots, A_{i-1} has invertible initials. Consider the polynomial A_i . If its initial is invertible then we continue our process, since such differential chains have good properties as we obtained in the previous section.

Suppose I_{A_i} is not invertible. Consider the ideal $J = (A_1, \dots, A_{i-1}) \subseteq k(Y)[U] = R$ for the corresponding sets Y and U . The ideal J may appear not to be a radical ideal. Let $J' = \sqrt{J}$. The rings R/J and R/J' have the same zero-divisors not taking into account the nilpotents of R/J . If I_{A_i} is a nilpotent of R/J then $I_{A_i} \in I$. Applying the autoreduction to $\mathbb{A} \cup I_{A_i}$ we obtain a differential chain with lower rank.

Let I_{A_i} be a non-nilpotent in R/J . Then I_{A_i} is a non-zero non-invertible element of R/J' . Since the initials of A_j are invertible, $1 \leq j \leq i - 1$, we obtain that $\dim J' = 0$. Thus, we have the minimal decomposition of J' into maximal ideals: $J' = \bigcap M_i$. Moreover, $R/J' \cong \bigoplus R/M_i$. Hence, the set of zero divisors of R/J' is equal to the set of non-invertible elements of

R/J' . Consider the polynomial A_i as a univariate polynomial in u_{A_i} . We have $A_i = I_{A_i}u_{A_i}^{n_i} + a_1u_{A_i}^{n_i-1} + \dots + a_{n_i}$ with $I_{A_i}, a_j \in R$ for $1 \leq j \leq n_i$.

Suppose that some a_j is invertible in R/J' . Then we can multiply A_i by the polynomial g corresponding to I_{A_i} such that $g \notin J'$ and $gI_{A_i} \in J'$ and reduce the rank of \mathbb{A} . The polynomial f is not invertible in R/J' iff $a \in M_l$ for some l . For any polynomial g we denote by $I(g)$ the set of all indices of M_i such that $g \in M_i$.

Generalizing the previous paragraph, we assert that if $I(I_{A_i})$ is not contained in $I(a_j)$ for each $1 \leq j \leq n_i$ then we can reduce the rank of \mathbb{A} multiplying A_i by the corresponding polynomial using the Chinese remainder theorem. Suppose that $I(I_{A_i})$ is strictly included in some $I(a_j)$. Consider the polynomial g corresponding to a_j such that $ga_j \in I$ and $I(g) \cap I(a_j) = \emptyset$. Note that $gA_i \notin I$. Reducing gA_i w.r.t. I we annihilate a_j . So, we may suppose that $I(I_{A_i}) = I(a_j)$ for $1 \leq j \leq n_i$.

Theorem 31. *Let I_{A_i} and a_j have the same sets of indices $I(I_{A_i})$ and $I(a_j)$ for $1 \leq j \leq n_i$. Then we can represent A_i as $g \cdot (I_{new}u_{A_i}^{n_i} + b_1u_{A_i}^{n_i-1} + \dots + b_{n_i})$, where g is not invertible and I_{new} depends only on non-leaders.*

Proof. Let $\bar{g} \in R/I$. Then $(\bar{g})_{R/I} = k_{i_1} \oplus \dots \oplus k_{i_m}$, where \bar{g} has 1 in the coordinates corresponding to k_{i_j} and 0 in the others. By the assumption all coefficients of A_i belong to the same set $\{k_{i_j}\}$ of fields. Consider the surjective ring homomorphism $\varphi : R \rightarrow R/I \cong \bigoplus k_i = R'$. Let $\overline{I_{A_i}} = \varphi(I_{A_i})$.

Since $I(I_{A_i}) = I(a_j)$, we know that $\overline{a_j} = \varphi(a_j)$ is divisible by $\overline{I_{A_i}}$ in R' , that is $\varphi(a_j) = \overline{h_j} \cdot \overline{I_{A_i}}$ for some h_j , such that $\overline{h_j} = \varphi(h_j)$. So, we represent $A_i = g \cdot (u_{A_i}^{n_i} + h_1u_{A_i}^{n_i-1} + \dots + h_{n_i})$ in $k(Y)[U]$. Clearing out denominators we obtain the final representation. \square

Since one cannot reduce the rank of a characteristic set, we know that if one of its initials is not invertible then it must be a common multiple of the corresponding element of the characteristic set.

Let $\mathbb{A} = A_1, \dots, A_m \subseteq k\{y_1, \dots, y_l\}$ be an autoreduced set. Let y_i be the leader of A_i for $1 \leq i \leq m$. We may consider the solution of the system of differential equations $A_i = 0$, $1 \leq i \leq m$, in the field extension $k \subseteq Qk\{y_{m+1}, \dots, y_l\}$ according to the following construction from commutative algebra. Let $R = k[x_1, \dots, x_n]$ and $I \subseteq R$ be the ideal generated by a system of algebraic equations $f_i = 0$, $1 \leq i \leq m$, i.e. $I = (f_1, \dots, f_m)$. Let A be a k -algebra. Then there is a bijection between the solutions of $f_i = 0$, $1 \leq i \leq m$, in A and the set of homomorphisms $\phi : R/I \rightarrow A$.

Both k and $Qk\{y_{m+1}, \dots, y_l\}$ are k -algebras and we can define the solution in terms of homomorphisms in these cases. Determining the solution in differential algebra we use a semi-universal field extension of k to give a solution for every differential equation in k . Thus, a solution in the extension $k \subseteq Qk\{y_{m+1}, \dots, y_l\}$ is helpful for us, too. Furthermore, the ring $k(Y)[U]$ is

more convenient for establishing some properties of ideals in it. In $k(Y)[U]/I$ we can operate with invertible polynomials as the elements of the field $k(Y)$. These are the reasons for using $k(Y)[U]$ in our computations. In the next section we extend the class of characterizable radical differential ideals to a new class of definable ideals. We establish characteristic properties of such ideals.

3. DEFINABLE RADICAL DIFFERENTIAL IDEALS

3.1. Introduction. We introduce the class of definable radical differential ideals. Each definable ideal has a uniquely determined characteristic set in some sense (Corollary 36).

Definition 32. We say that a radical differential ideal I is *definable* iff I has a characteristic set in Kolchin's sense with invertible initials and separants.

It is clear that every characterizable ideal is definable. Consider the following example.

Example 33. Consider the ring of differential polynomials in one variable $k\{y\}$. As we have already seen the ideal $I = \{y'^2 + y\}$ is not characterizable but $A = y'^2 + y$ has an invertible separant, since A is a square-free polynomial.

The property of an ideal to be definable can be checked algorithmically as we see below. A characteristic set being obtained, in order to investigate the property of an ideal to be definable we should check, if there is a characteristic set of I with invertible initials and separants. So, we have a problem of inverting initials if we have a characteristic set with non-invertible initials. If we cannot invert these initials then I is not a definable ideal. To prove the final result we need to consider the following lemma from commutative algebra.

Lemma 34. *Let I be an ideal in the polynomial ring $k[x_1, \dots, x_{n+1}]$ generated by polynomials from $k[x_1, \dots, x_n]$ and suppose $f = a_m x_{n+1}^m + \dots + a_1 x_{n+1} + a_0 \in I$, where $a_i \in k[x_1, \dots, x_n]$, $0 \leq i \leq m$. Then $a_i \in I$, $0 \leq i \leq m$.*

Proof. Let $I = (f_1, \dots, f_k)$, $f_i \in k[x_1, \dots, x_n]$, $0 \leq i \leq k$. Represent f as an element of the ideal I : $f = g_1 f_1 + \dots + g_k f_k$, $g_i \in k[x_1, \dots, x_{n+1}]$, $0 \leq i \leq k$. Considering this expression as a polynomial in x_{n+1} we conclude the statement of the lemma. \square

Proposition 35. *Suppose $\mathbb{A} = A_1, \dots, A_n$ is a characteristic set of a radical differential ideal I with invertible initials, and there exists i , $1 \leq i \leq n$, such that S_{A_i} is not invertible. Then I is not a definable ideal.*

Proof. Suppose I is a definable ideal and $\mathbb{C} = C_1, \dots, C_n$ is a characteristic set of I with invertible initials and separants. Since the initials of \mathbb{C} and

\mathbb{A} are invertible, we may assume that they depend only on non-leaders. Indeed, to obtain new \mathbb{C}' and \mathbb{A}' we only need to multiply them by invertible reduced polynomials and perform a reduction. The separants of \mathbb{C}' appear to be invertible. We note an important property of a characteristic set of an arbitrary differential ideal in this situation: $I_{A_i}C_i = I_{C_i}A_i$. Indeed, let $A_i = I_{A_i}u_{A_i}^m + a_1u_{A_i}^{m-1} + \dots + a_m$ and $C_i = I_{C_i}u_{A_i}^m + c_1u_{A_i}^{m-1} + \dots + c_m$, where u_{A_i} is the leader of A_i .

We have $I_{A_i}C_i - I_{C_i}A_i = \sum_{j=1}^m (c_jI_{A_i} - a_jI_{C_i})u_{A_i}^{m-j} \in I$. We shall call this polynomial the *I-polynomial* of C_i and A_i . It is reducible to zero w.r.t. \mathbb{A} . Using Lemma 34, we obtain $c_jI_{A_i} = a_jI_{C_i}$, $1 \leq j \leq m$, because the initials depend only on the non-leaders. According to Proposition 20, f is not invertible w.r.t. \mathbb{A} iff there exists a polynomial g such that $fg \in (A)$ and g is not reducible to zero w.r.t. \mathbb{A} . We have $S_{A_i} = mI_{A_i}u_{A_i}^{m-1} + (m-1)a_1u_{A_i}^{m-2} + \dots + a_{m-1}$, $S_{C_i} = mI_{C_i}u_{A_i}^{m-1} + c_1u_{A_i}^{m-2} + \dots + c_{m-1}$. Hence, $I_{C_i}S_{A_i} = I_{A_i}S_{C_i}$ and $I_{C_i}S_{A_i}$ is invertible, thus S_{A_i} is invertible. We obtain a contradiction. \square

Corollary 36. *According to this proof we can suppose that the initials of a characteristic set with invertible initials depend only on non-leaders. I-polynomials being considered, we obtain that these characteristic sets are equal in $k(Y)[U]$.*

Remark 37. It should be noticed that Proposition 35 is a partial case of Proposition 26. We presented a special proof of Proposition 35 using only the basic notion of characteristic set. The proof of Proposition 26 is based on a technique developed in commutative algebra, e.g. localization, rings of Cohen-Macaulay.

So, a characteristic set with invertible initials of a definable ideal is uniquely determined in the ring of quotients and we have a one-to-one correspondence between definable ideals and their characteristic sets with invertible initials. These characteristic sets must have invertible separants (Proposition 35).

3.2. Characteristic Sets of Definable Radical Differential Ideals.

Theorem 38. *Suppose \mathbb{A} is a characteristic set of an ideal I and some initial of \mathbb{A} is not invertible. Then we can check, whether I has a characteristic set \mathbb{C} with invertible initials.*

Proof. Suppose that I has a characteristic set $\mathbb{C} = C_1, \dots, C_n$ with invertible initials. Multiplying C_i , $1 \leq i \leq n$, by corresponding polynomials, we obtain a characteristic set with initials depending only on non-leaders. Suppose A_i is the smallest polynomial in \mathbb{A} with non-invertible initial. We have $i \geq 2$. Consider the I-polynomial $\sum_{j=1}^m (c_jI_{A_i} - a_jI_{C_i})u_{A_i}^{m-j}$ in $k(Y)[U]$. Using Lemma 34 and remembering that the product of partially reduced polynomials is a partially reduced polynomial, we have $(c_jI_{A_i} - a_j) \in (A_1, \dots, A_{i-1}) : I_{A_i}^{-\infty}$. We really dismiss differentiations and clearly have a commutative algebraic construction.

We suppose $c_0 = I_{C_i} = 1$. These conditions are necessary and sufficient for the existence of a characteristic set with invertible initials. The last formula checks the inclusion in the fractional ideal. To do this work we can use the Gröbner bases technique with parameters — the coefficients of the new characteristic set \mathbb{C} . Using the property that our system is autoreduced, we have restrictions on the degrees of the coefficients. So, we only need to find the solution of the system of algebraic equations. \square

Example 39. Consider the autoreduced set $\mathbb{A} = x^2 + x, xy + x$ in $k\{x, y\}$, $x < y$. The initial x is not invertible. Let I be the ideal $\{x^2 + x, y - x\}$ in $k\{x, y\}$. We have $\mathbb{A} \subseteq I$. The second polynomial $y - x$ has an invertible initial. Hence, this ideal is evidently definable. So, \mathbb{A} is a characteristic set of I . We try to obtain a characteristic set \mathbb{C} with invertible initials using the characteristic set \mathbb{A} . The second polynomial can be represented as $y + (cx + b)$, where $c, b \in k(Y)$.

We have $(I_{A_2}(cx + b) - x) \in (x^2 + x) : 1^\infty \iff x(cx + b) - x = a(x^2 + x) \iff cx^2 + (b - 1)x = ax^2 + ax$. Hence, we obtain $c = a$ and $b = c + 1$. So, a characteristic set with invertible initials should equal $x^2 + x, y + cx + c + 1$, where $c \in k(Y)$. If some element of this family belongs to I , we choose it as a characteristic set with invertible initials. Choosing $c = -1$, we easily obtain that I is a definable ideal. On the other hand, the ideal generated by \mathbb{A} is not definable by the same reason. It is easy to see that there are no elements in this family belonging to $\{\mathbb{A}\}$. Hence, $\{x^2 + x, xy + x\}$ is not a definable ideal.

Remark 40. Example 39 shows that there exists a definable ideal $I = \{x^2 + x, y - x\}$ with two “absolutely” different characteristic sets.

According to Proposition 35 and Theorem 38 to check whether a radical differential ideal I is definable, we should compute its characteristic set \mathbb{C} , invert the initials and check, whether the separants of \mathbb{C} are invertible. As a conclusion of the previous discussions we obtain the following result.

Theorem 41. *A radical differential ideal I is definable iff it has a characteristic set with invertible initials and this set must have invertible separants.*

4. CONSEQUENCES

The proved theorems and the examples show that the property of a radical differential ideal to be characterizable is non-trivial. If you look at the ideal and keep your attention on it for a few minutes, probably you will not solve this problem for the particular task. Nevertheless, in many cases a characteristic decomposition of the radical differential ideal and the criteria obtained in this paper help to decide whether it is characterizable. It should be noticed that Theorem 41 allows to check whether some ideal is definable algorithmically if we know some characteristic set of this ideal. We have investigated a part of this problem and presented a general idea in this field, thus covering some problems of constructing a characteristic set of a radical differential ideal.

There is the very interesting problem of how to obtain a characteristic set of every radical differential ideal (in particular, definable ideal). Yet another very natural problem is to generalize the presentation of a characteristic set of a radical differential ideal given in Theorem 31.

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