APPLICATIONS ORIENTED MATHEMATICS
PRELIMINARY EXAMINATION

Friday, April 30, 2004 1:00-4:00pm

Each of the 8 numbered questions is worth 20 points. All questions will be graded, but your score for the examination will be the sum of the scores of your best **FIVE** questions.

Use a separate answer booklet for each question, and do not put your name on the answer booklets, instead put the number that is on the envelope the exam was in. When you have completed the examination, insert all your answer booklets in the envelope provided. Then seal and print your name on the envelope.

1. Find the leading term in the asymptotic expansion of the following integral in the indicated limits

$$\int_0^1 \sqrt{t(1-t)}\exp(xt(1-t))dt.$$

(a) $x \to \infty$ (b) $x \to -\infty$.

2. Consider the following integral as $\lambda \to +\infty$

$$I(\lambda) = \int_{-\infty}^{\infty} e^{\lambda(it - t^4/4)}dt.$$

   (a) Find all saddle points in the complex t plane.
   (b) At each saddle, find the local steepest descent (SD) and steepest ascent (SA) directions.
   (c) Find the (global) steepest descent contour though each saddle.
   (d) Find the leading term in the asymptotic expansion of the integral.

3. Consider the ODE

$$y''(x) = (x^4 + x)y(x).$$

Find the leading term in the expansions of all solutions as $x \to \infty$ (show your work!).

4. Consider the sums

(a) $\sum_{k=1}^{n} e^{k^2}$ (b) $\sum_{k=1}^{n} e^{k}$ (c) $\sum_{k=1}^{n} e^{\sqrt{k}}$.

In each case find the leading term as $n \to \infty$.

5. Use singular perturbation methods to construct a leading order composite or uniform approximation to $y(x)$ as $\varepsilon \to 0^+$

$$\varepsilon y'' + x^2 y' = 0, \quad y(-1) = 3, \quad y(1) = 1.$$

Locate all boundary layers and indicate their thickness.

6. Find the leading order approximation for $\varepsilon \to 0^+$

$$\varepsilon \Delta u + u_y = 0 \ \text{ in } x > 0, \ \ 0 < y < 1$$

with the boundary conditions

$$u(x,0) = 0, \ \ x > 0, \quad u(x,1) = 2 - e^{-x}, \ \ x > 0, \ \ u(0,y) = 0, \ \ 0 < y < 1$$

(Hint: To solve a parabolic equation of the form $u_{\eta\eta} + u_y = 0$ in $0 < y < 1$, $0 < \eta < \infty$ try a similarity variable $\eta/\sqrt{1-y}$.)

7. Use a perturbation method to find the first 2 terms in the expansion of the eigenvalues and eigenvectors, for the perturbed eigenvalue problem

$$\mathbf{A}x = \lambda x, \ < x, x >= 1, \ \ \mathbf{A} = \begin{pmatrix} 1 & 1-\varepsilon^2 \\ \varepsilon^2 & 1 \end{pmatrix}$$

8. Find the leading term (valid for large t) in the asymptotic expansion of $x(t)$ using a multiple scales method

$$x'' + x + \varepsilon(x')^3 = 0, \quad x(0) = 1, \ \ x'(0) = 0.$$